MATH 1A FINAL (PRACTICE 1)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Calculate the following (you do not need to use the limit definition):

(a) (10 points)

$$\frac{d}{dx}\ln(3x)e^{2x}$$

Solution:

$$\frac{d}{dx}\ln(3x)e^{2x} = \frac{3}{3x}e^{2x} + \ln(3x)\cdot 2e^{2x}$$

(b) (15 points)

$$\frac{d}{dx}\tan(e^{(x^x)})$$

Solution:

$$f(x) = x^x \Rightarrow \ln(f(x)) = x\ln(x) \Rightarrow \frac{d}{dx}\ln(f(x)) = \ln(x) + 1$$

$$\Rightarrow f'(x) = x^x(\ln(x)+1)$$

$$\Rightarrow \frac{d}{dx}\tan(e^{(x^x)}) = \sec^2(e^{(x^x)})\cdot e^{(x^x)}\cdot x^x(\ln(x)+1)$$

2. Calculate the following (you do not need to use the (ϵ, δ)-definition):

(a) (10 points)

$$\lim_{x\to 0} \frac{\sin(x^2)}{\tan x}$$

Solution:

L'Hospital

$$\lim_{x\to 0} \frac{\sin(x^2)}{\tan(x)} = \lim_{x\to 0} \frac{2x\cos(x^2)}{\sec^2(x)} = \frac{\lim_{x\to 0} 2x\cos(x^2)}{\lim_{x\to 0} \sec^2(x)}$$

$$= \frac{0}{1} = 0$$

(b) (15 points)

$$\lim_{x\to -\infty} \frac{\sqrt{1+4x^6}}{2+x^3}$$

Solution:

$$\lim_{x\to -\infty} \frac{\sqrt{1+4x^6}}{2+x^3} = \lim_{x\to -\infty} \frac{\left(\frac{\sqrt{1+4x^6}}{x^3}\right)}{\left(\frac{2+x^3}{x^3}\right)}$$

$$= \lim_{x\to -\infty} \frac{\left(\frac{\sqrt{1+4x^6}}{-\sqrt{x^6}}\right)}{\frac{2}{x^3}+1} = \lim_{x\to -\infty} \frac{-\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}+1}$$

$$= \frac{-\sqrt{4}}{1} = -2$$

3. Calculate the following (you do not need to use Riemann sum defintion):

(a) (10 points)

$$\int \frac{2x+4}{x^2+1}dx$$

Solution:

$$\int \frac{2x+4}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{4}{x^2+1}\,dx$$

$$u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow \int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln|x^2+1| + C$$

$$\Rightarrow \int \frac{2x+4}{x^2+1}\,dx = \ln|x^2+1| + 4\arctan(x) + C$$

(b) (15 points)

$$\int_e^{e^2} \frac{1}{x\ln(x)}dx$$

Solution:

$$u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$$

$$\Rightarrow \int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|\ln(x)| + C$$

$$\Rightarrow \int_e^{e^2} \frac{1}{x\ln(x)}\,dx = \ln(2) - \ln(1) = \ln(2)$$

4. (25 points) A cup of water is placed in a refrigerator. The refrigerator has temperature 5C. After 10 minutes the water is 15C. After 20 minutes the water is 10C. Determine the temperature of the water when it was placed in the refrigerator.

Solution:

Initial temperature

$$\text{NLC} \Rightarrow T(t) = T_s + (T_0 - T_s)e^{kt}$$

$$T(10) = 15 \Rightarrow 15 = 5 + (T_0 - 5)e^{10k}$$

$$T(20) = 10 \Rightarrow 10 = 5 + (T_0 - 5)e^{20k}$$

$$\Rightarrow \frac{10}{T_0 - 5} = e^{10k} \quad \underline{\text{and}} \quad \frac{5}{T_0 - 5} = e^{20k}$$

$$\Rightarrow \frac{10^2}{(T_0 - 5)^2} = \frac{5}{(T_0 - 5)} \Rightarrow 20 = T_0 - 5$$

$$\Rightarrow T_0 = 25^\circ C$$

5. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \frac{x^2e^x}{x}$$

Solution:

$$f(x) = \frac{x^2e^x}{x} = \begin{cases} xe^x & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases}$$

Sketch $y = xe^x$ and remove $(0,0)$

<u>Domain</u> : $\mathbb{R}$

<u>Odd/Even</u> : Neither

<u>Vertical Asymptotes</u> : None

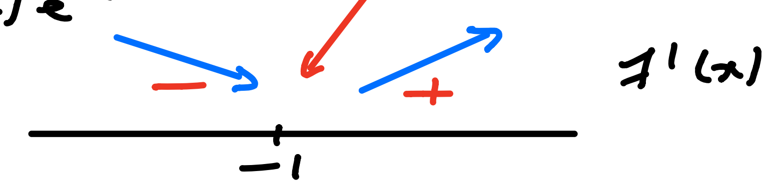
$$\lim_{x\to\infty} xe^x = \infty$$

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0$$

$\Rightarrow$ $y=0$ horizontal asymptote as $x \to -\infty$

f chg at -1 $\Rightarrow$ local min

$$f'(x) = e^x + xe^x = (1+x)e^x$$



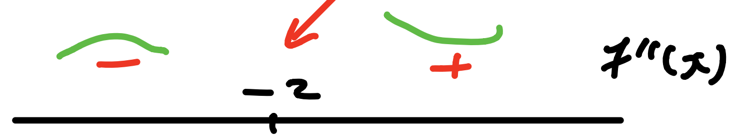
A/ $f'(x) = 0 \iff x = -1$

B/ None

Solution (continued) :

$$f''(x) = e^x + (1+x)e^x = (2+x)e^x$$

f cts at -2
$\Rightarrow$ inflection

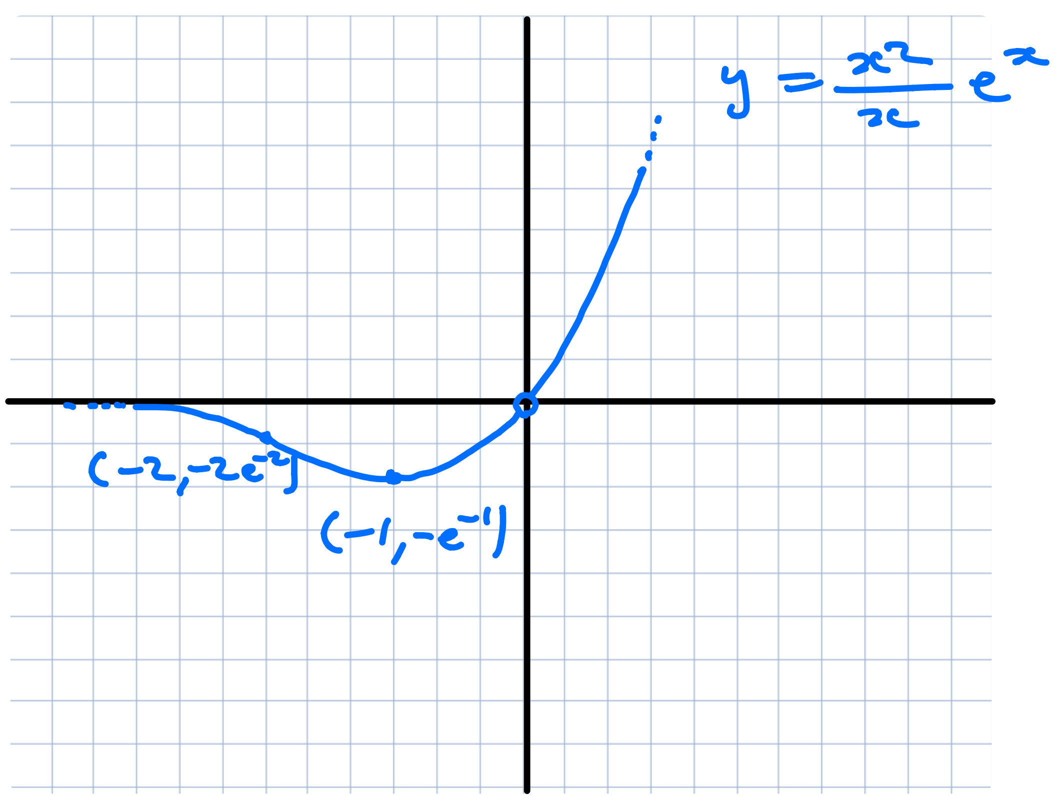


A/ $f''(x) = 0 \iff x = -2$

B/ None

$$f(-1) = -e^{-1}$$

$$f(-2) = -2e^{-2}$$



6. (25 points) Determine the equation of the tangent line at $x = 1$ of the following curve:

$$y = \int_x^{3x} \cos(\pi t)dt + 2x$$

Solution:

$$y = \int_0^{3x} \cos(\pi t)\,dt - \int_0^{x} \cos(\pi t)\,dt + 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = 3\cos(3\pi x) - \cos(\pi x) + 2$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=1} = 3\cos(3\pi) - \cos(\pi) + 2$$

$$= -3 - (-1) + 2 = 0$$

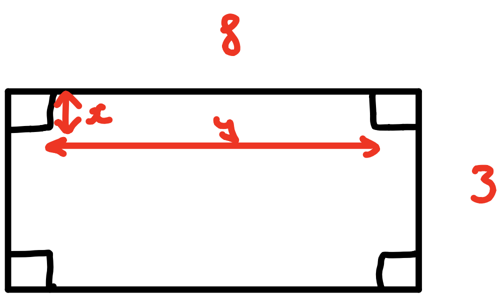
$$\int_1^3 \cos(\pi t)\,dt + 2 = \frac{1}{\pi}\sin(\pi t)\Big|_1^3 + 2 = 0 + 2 = 2$$

$\Rightarrow$ Tangent line has equation $y = 2$

7. (25 points) An open box will be made by cutting a square from each corner of a 3 by 8 foot piece of cardboard and then folding up the sides. What size squares should be cut from each corner to maximize the volume.

Solution:

Objective : Maximize Volume



Objective : Volume $= y \cdot x \cdot (3-2x)$

Constraint : $2x + y = 8$

$\Rightarrow \quad y = 8 - 2x$

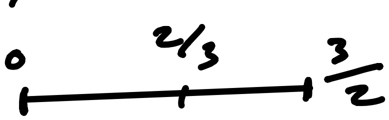
$\Rightarrow$ Volume $= (8-2x)\, x\, (3-2x) = +4x^3 - 22x^2 + 24x$
$= f(x)$

Domain : $[0, \frac{3}{2}]$

$f'(x) = 12x^2 - 44x + 24 = 3x^2 - 11x + 6$
$= (3x-2)(x-3)$

A) $f'(x) = 0 \Rightarrow x = 2/3$ or 3

B) f' continuous everywhere



$f(0) = 6$

$f(3/2) = 0$

$f(2/3) > 0$

$\Rightarrow$ Volume is maximized when $x = 2/3$ (8 inches)

8. (25 points) A company accrues debt at a rate of

$$(2t+3)\sqrt{t+1}$$

dollars per year, where t is the time in years since the company started. How much will the company's debt have grown between $t = 3$ and $t = 8$? You do not need to simplify your answer.

Solution:

$$D'(t) = (2t+3)\sqrt{t+1}$$

$$u = t+1 \Rightarrow \frac{du}{dt} = 1 \Rightarrow dt = du$$

$(\Rightarrow t = u-1)$

$$\Rightarrow \int (2t+3)\sqrt{t+1}\, dt = \int (2t+3)\sqrt{u}\, du$$

$$= \int (2(u-1)+3)\sqrt{u}\, du = \int 2u^{3/2} + u^{1/2}\, du$$

$$= 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \frac{4}{5}(t+1)^{5/2} + \frac{2}{3}(t+1)^{3/2} + C$$

$$\Rightarrow \int_3^8 (2t+3)\sqrt{t+1}\, dt = \frac{4}{5}(t+1)^{5/2} + \frac{2}{3}(t+1)^{3/2}\Big|_3^8$$

$$= \left(\frac{4}{5}\cdot 3^5 + \frac{2}{3}3^3\right) - \left(\frac{4}{5}2^5 + \frac{2}{3}2^3\right)$$

$$= D(8) - D(3)$$

9. (25 points) Calculate the following limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6n+3i}{n^2}\sin((2+\frac{i}{n})^2)$$

Solution:

$$\frac{6n+3i}{n^2}\sin((2+\frac{i}{n})^2) = 3\cdot(2+\frac{i}{n})\sin((2+\frac{i}{n})^2)\cdot\frac{1}{n}$$

Let $a=2$, $b=3$ and $f(x) = 3x\sin(x^2)$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{6n+3i}{n^2}\sin((2+\frac{i}{n})^2) = \int_2^3 3x\sin(x^2)\,dx$$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\Rightarrow \int 3x\sin(x^2)\,dx = \frac{3}{2}\int \sin(u)\,du = \frac{-3}{2}\cos(u)+C$$
$$= \frac{-3}{2}\cos(x^2)+C$$

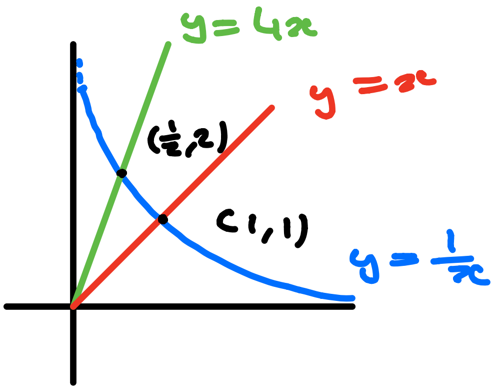
$\Rightarrow$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6n+3i}{n^2}\sin((2+\frac{i}{n})^2) = \frac{-3}{2}\cos(x^2)\Big|_2^3$$
$$= \frac{3}{2}(\cos(4)-\cos(9))$$

10. Let $f(x) = 1/x$ and $g(x) = 4x$ $h(x) = x$.

(a) (10 points) Calculate the area of the finite region bounded by $y = f(x)$, $y = g(x)$ and $y = h(x)$.

Solution:



$$\text{Area} = \int_0^{\frac{1}{2}} 4x - x \, dx + \int_{1/2}^{1} \frac{1}{x} - x \, dx$$

$$= \frac{3}{2}x^2 \Big|_0^{1/2} + \ln|x| - \frac{1}{2}x^2 \Big|_{1/2}^{1}$$

$$= \frac{3}{8} + \left(\left(-\frac{1}{2}\right) - \left(\ln\left(\frac{1}{2}\right) - \frac{1}{8}\right)\right)$$

$$= \ln(2)$$

(b) (15 points) Calculate the volume of the solid of revolution formed by rotating this region around the x-axis.

Solution:

$$\text{Volume} = \pi \int_0^{1/2} (4x)^2 - x^2 \, dx + \pi \int_{1/2}^{1} \frac{1}{x^2} - x^2 \, dx$$

$$= 5\pi x^3 \Big|_0^{1/2} + \pi \left(\frac{-1}{x} - \frac{1}{3}x^3\right)\Big|_{1/2}^{1}$$

$$= \frac{5\pi}{8} + \pi \left(\left(-1 - \frac{1}{3}\right) - \left(-2 - \frac{1}{24}\right)\right)$$

END OF EXAM