Exponential Functions

$b > 0$

$f(x) = b^x$ ← b = base, x = exponent

$n > 0$ an integer $\Rightarrow$ $b^n = \underbrace{b \cdot b \cdots b}_{n \text{ times}}$

$$b^{-n} = \frac{1}{b^n} = \left(\frac{1}{b}\right)^n$$

$b^0 = 1$

m integer $\Rightarrow$ $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$ ← rational exponent

If x irrational ← x not a fraction

more complicated

Example

irrational

$\sqrt{3} = 1.73205\ldots$

We define $2^{\sqrt{3}}$ to be the number that the following sequence gets closer and closer to:

$$2^1, \quad 2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \ldots$$

$$2^1 \qquad 2^{\frac{17}{10}} \qquad 2^{\frac{173}{100}} \qquad 2^{\frac{1732}{1000}}$$

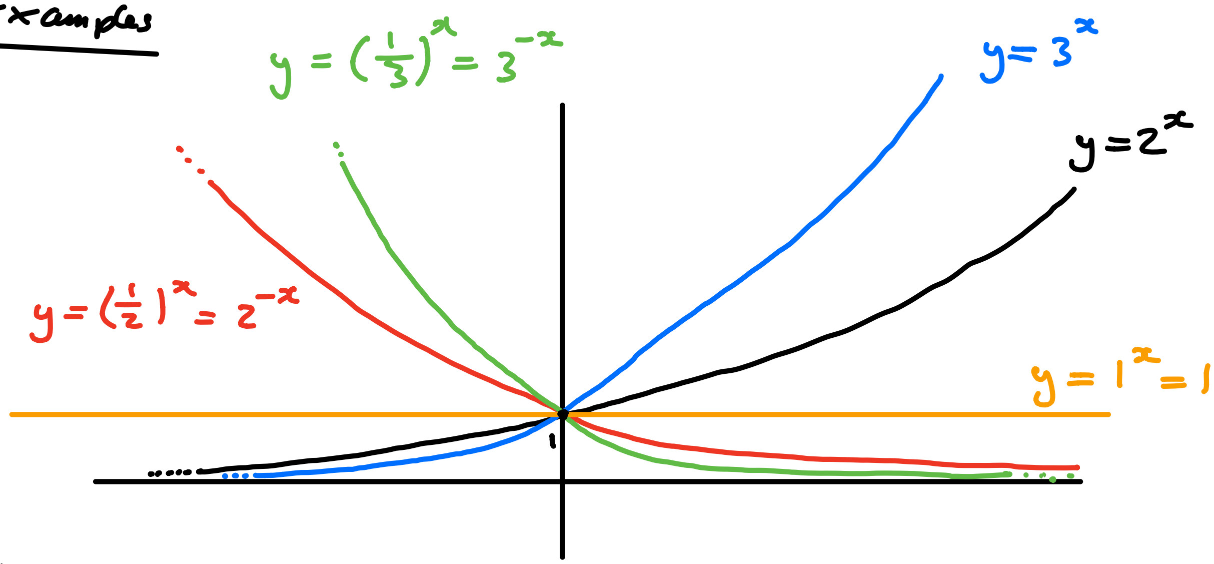
$2^{\sqrt{3}} = 3.32199\ldots$

Fact: This works for any $b > 0$ and x in $\mathbb{R}$

$\Rightarrow$ Domain of $f(x) = b^x$ is $\mathbb{R}$

Range of $f(x) = b^x$ is $(0, \infty)$

Examples



$\Rightarrow$

$b > 1 \Rightarrow f(x) = b^x$ increasing on $\mathbb{R}$

$b < 1 \Rightarrow f(x) = b^x$ decreasing on $\mathbb{R}$

Laws of Exponents

1/ $b^{(x+y)} = b^x \cdot b^y$ 2/ $b^{(x-y)} = \dfrac{b^x}{b^y}$

3/ $b^{(xy)} = (b^x)^y$ 4/ $(ab)^x = a^x b^x$ $\leftarrow a, b > 0$

Observation

$b^{-x} = \left(\frac{1}{b}\right)^x \Rightarrow y = \left(\frac{1}{b}\right)^x$ is reflection of $y = b^x$ in y-axis

Practical Examples

1/ (Natural Population Growth)

A population with initial size P_0 triples in size in any yearly interval.

Mathematical Model :

t = time in years after population is P_0

$P(t)$ = population size at time t

$P(0) = P_0$

$P(1) = P_0 \cdot 3$

$P(2) = P_0 \cdot 3 \cdot 3 = P_0 \cdot 3^2$

$P(3) = P_0 \cdot 3^2 \cdot 3 = P_0 \cdot 3^3$

More generally : $P(t) = P_0 3^t$ for t in $[0, \infty)$

Check :

- $P(0) = P_0$ ✓
- $P(t+1) = P_0 3^{t+1} = P_0 3^t \cdot 3 = P(t) \cdot 3$ ✓

2/ A radioactive substance with initial mass M_0, loses half its mass in any yearly interval.

$$M(t) = M_0 \left(\frac{1}{2}\right)^t = M_0 2^{-t}$$

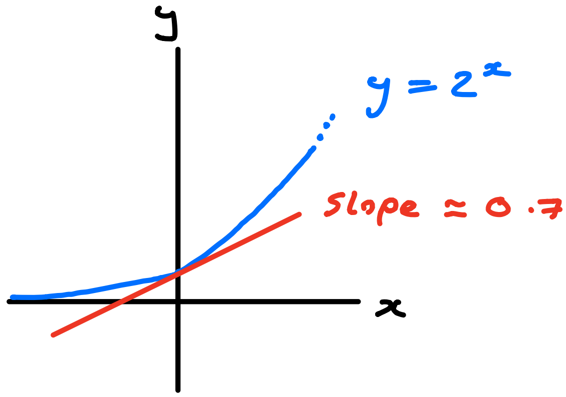
time after mass is M_0

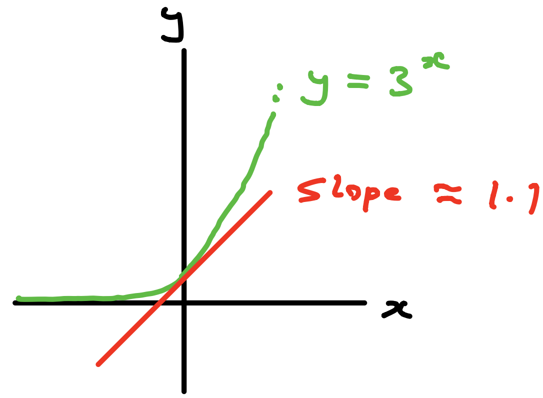
Observation

approximately equal

Slope at $y = 2^x$ at $x = 0 \approx 0.7$

Slope at $y = 3^x$ at $x = 0 \approx 1.1$

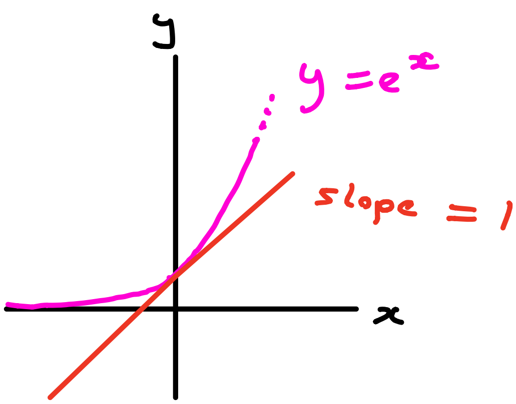




some fixed real number

$\Rightarrow$ There exists $2 < e < 3$ such that

slope at $y = e^x$ at $x = 0$ is precisely 1



$e = 2.718281828\ldots$

$f(x) = e^x$ "natural exponential function"

Discovered by Euler in 1727