Derivatives and Graph Shape

Sign Analysis : When is a function positive or negative?

Let g be a function. Assume $g(x)$ changes sign at $x = c$, ie

$$\underset{g(x)>0}{+} \quad c \quad \underset{g(x)<0}{-} \qquad \text{or} \qquad \underset{g(x)<0}{-} \quad c \quad \underset{g(x)>0}{+}$$

Intermediate Value Theorem $\Rightarrow$

A/ $g(c) = 0$ or B/ g discontinuous at $x = c$ (e.g. $g(c)$ undefined)

Algorithm to determine where g is + or −

1/ Find all points c such that A/ or B/ holds.

2/ Draw number line and mark all c from 1/.

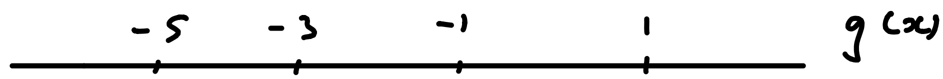
3/ Evaluate $g(x)$ at test points between and around these c. Their sign tells us about the sign of g everywhere.

Example $g(x) = \dfrac{(x+1)(x-1)}{(x+3)(x+5)}$

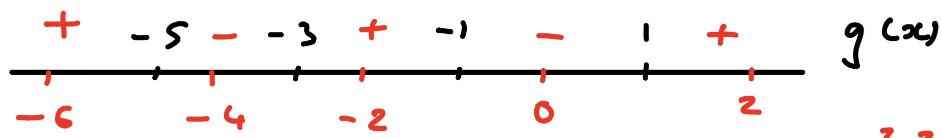
1/ A/ $g(c) = 0 \iff c = -1$ or $c = 1$

B/ g discontinuous at $c \iff c = -3$ or $c = -5$

2/



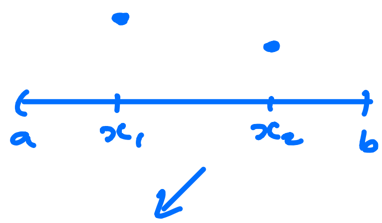
3/



$g(-6) = \frac{(-5)\cdot(-7)}{(-3)\cdot(-1)} > 0$ $\quad g(-2) = \frac{(-1)\cdot(-3)}{1\cdot 3} > 0$ $\quad g(2) = \frac{3\cdot 2}{5\cdot 7} > 0$

$g(-4) = \frac{(-3)\cdot(-5)}{(-1)\cdot 1} < 0$ $\quad g(0) = \frac{1\cdot(-1)}{3\cdot 5} < 0$

f - differentiable on (a,b)



Observations

f not increasing on $(a,b) \Rightarrow$ There exist $x_1 < x_2$ in (a,b) such that $f(x_1) \geq f(x_2)$

M.V.T.
$\Rightarrow$ There exists c in (a,b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0$$

Similarly,

f not decreasing on $(a,b) \Rightarrow$ There exists c in (a,b) such that $f'(c) \geq 0$

Conclusion :

Increasing-Decreasing Test

Assuming f differentiable on (a,b), then

1/ $f'(x) > 0$ on (a,b) $\Rightarrow$ f increasing on (a,b)

2/ $f'(x) < 0$ on (a,b) $\Rightarrow$ f decreasing on (a,b)

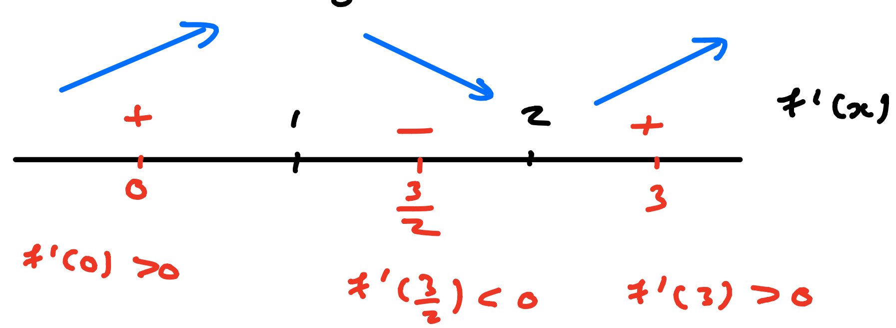
Example $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 1$

$f'(x) = x^2 - 3x + 2$

Sign-Analysis on $f'(x)$:

A/ $f'(x) = (x-2)(x-1) \Rightarrow f'(c) = 0 \Leftrightarrow c = 1, 2$

B/ f' continuous everywhere



$\Rightarrow$ f increasing on $(-\infty, 1)$ and $(2, \infty)$

f decreasing on $(1, 2)$

Observations :

f - continuous on (a,b) and $a < c < b$.

f increasing on (a,c)
and $\Rightarrow$ $f(c)$ local max
decreasing on (c,d)

f decreasing on (a,c)
and $\Rightarrow$ $f(c)$ local min
increasing on (c,d)

Conclusion :

First Derivative Test for Local Max/Min

Let f be continuous on (a,b) and $a<c<b$ such that f differentiable on (a,c) and (c,b). Then

$+$ c $-$ $f'(x)$ $\Rightarrow$ $f(c)$ a local max

$f'(x)>0$ $f'(x)<0$

$-$ c $+$ $f'(x)$ $\Rightarrow$ $f(c)$ a local min

$f'(x)<0$ $f'(x)>0$

In previous example $f(1)$ is a local max and $f(2)$ is a local min.

Strategy to find Local Max/Min of continuous f

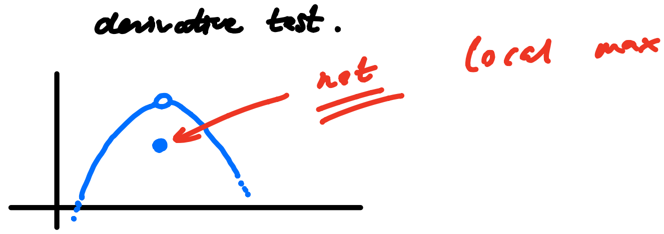
1/ Calculate f'.

2/ Do sign analysis on f'.

3/ Apply First Derivative test at critical numbers where f continuous.

WARNING: If f not continuous at c, cannot apply first derivative test.

E.g.



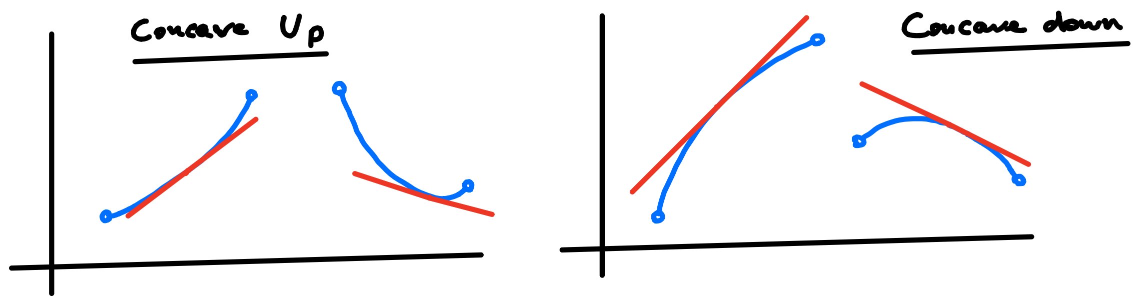
Definition

Let f be differentiable on (a,b).

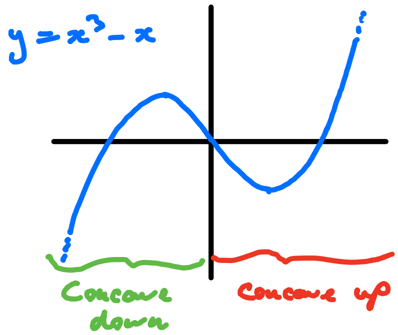
1/ We say f is concave up on (a,b) if, for each c in (a,b), $y=f(x)$ lies above tangent line at $x=c$.

2/ We say f is concave down on (a,b) if, for each c in (a,b), $y=f(x)$ lies below tangent line at $x=c$.

Basic Picture:



Example $f(x) = x^3 - x = x(x^2-1) = x(x+1)(x-1)$



$\Rightarrow$ f concave up on $(0, \infty)$

f concave down on $(-\infty, 0)$

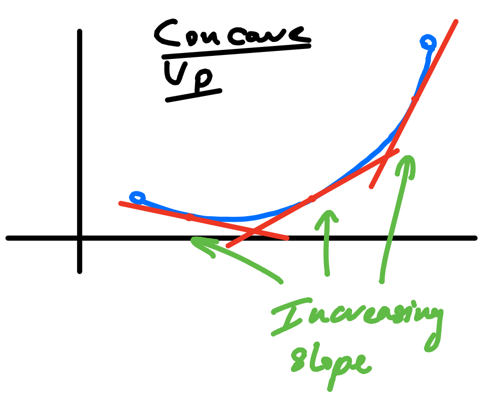
Definition

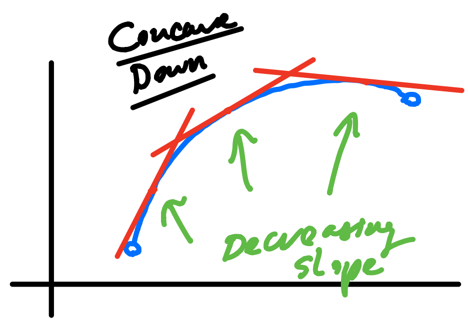
An inflection point at $(c, f(c))$ on the graph where

1/ f continuous at $x = c$

2/ The graph changes concavity at $(c, f(c))$.

Q/ : How can we determine the concavity of a function?





f' increasing on (a, b) $\Rightarrow$ f concave up on (a, b)

f' decreasing on (a, b) $\Rightarrow$ f concave down on (a, b)

Conclusion (Concavity Test)

$$\boxed{\begin{array}{l} f''(x) > 0 \text{ on } (a,b) \Rightarrow f' \text{ increasing on } (a,b) \Rightarrow f \text{ Concave up on } (a,b) \\ f''(x) < 0 \text{ on } (a,b) \Rightarrow f' \text{ decreasing on } (a,b) \Rightarrow f \text{ concave down on } (a,b) \end{array}}$$

Example $f(x) = x^3 - x \Rightarrow f'(x) = 3x^2 - 1 \Rightarrow f''(x) = 6x$

A/ $f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$

B/ f'' continuous everywhere

$f''(-1) = 6\cdot(-1) = -6 < 0$ $\quad f''(1) = 6\cdot 1 = 6 > 0$

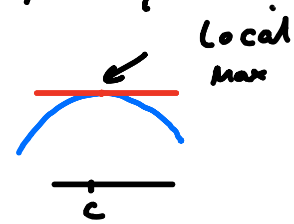
$\Rightarrow$ f concave down on $(-\infty, 0)$.

f concave up on $(0, \infty)$

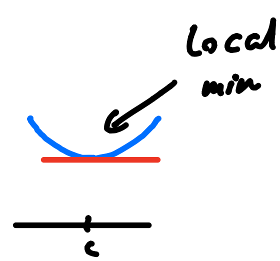
f continuous at 0 $\Rightarrow$ $(0,0)$ is an inflection point on $y = x^3 - x$

Q/: Can we use concavity to find local max/min points?

$f'(c) = 0$
$f''(c) < 0$ $\Rightarrow$ Tangent line horizontal at $x = c$ and Concave down at $x = c$ $\Rightarrow$



$f'(c) = 0$
$f''(c) > 0$ $\Rightarrow$ Tangent line horizontal at $x = c$ and Concave up at $x = c$ $\Rightarrow$



Second Derivative Test for Local Max/Min

Let f'' exist on some open interval containing c. Then
(except maybe at c)

1/ $f''(c) > 0$ and $f'(c) = 0$ $\Rightarrow$ $f(c)$ local min

2/ $f''(c) < 0$ and $f'(c) = 0$ $\Rightarrow$ $f(c)$ local max

3/ If $f''(c) = 0$ or DNE we cannot conclude anything.

Example

$$f(x) = x^3 - x \Rightarrow f'(x) = 3x^2 - 1 \Rightarrow f''(x) = 6x$$

$$f'(x) = 0 \Leftrightarrow x = \frac{\pm 1}{\sqrt{3}}$$

$$f''\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} > 0 \Rightarrow f\left(\frac{1}{\sqrt{3}}\right) \text{ local min}$$

$$f''\left(\frac{-1}{\sqrt{3}}\right) = \frac{-6}{\sqrt{3}} < 0 \Rightarrow f\left(\frac{-1}{\sqrt{3}}\right) \text{ local max}$$