The Derivative as a Function

Derivative of f at $x = a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \text{Slope of tangent line at } x = a$$

Change in perspective : Treat a like variable x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow$$

f' is new function which we call the derivative of f.

Domain of f' = All x such that f differentiable at x

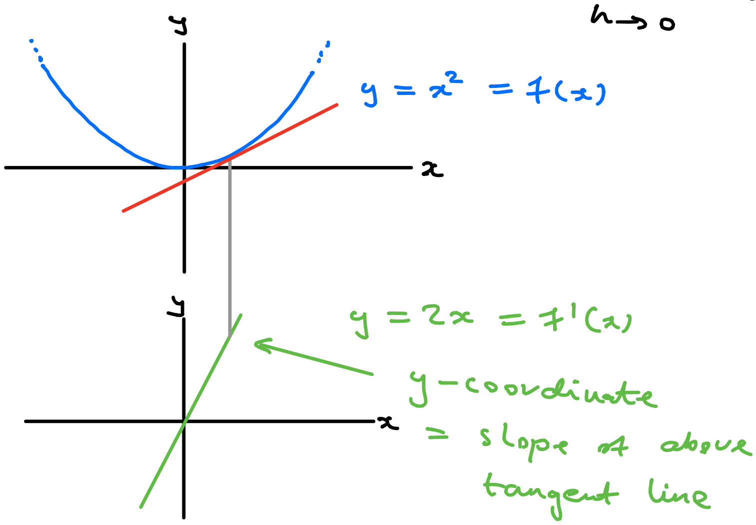
Example

1/ $f(x) = x^2 \Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h = 2x$$



2/ $$f(x) = \frac{1-x}{1+x} \Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

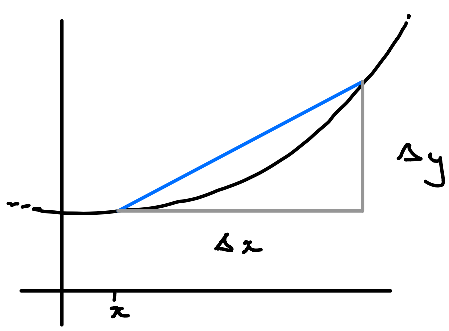
$$= \lim_{h \to 0} \frac{\frac{1-(x+h)}{1+(x+h)} - \frac{1-x}{1+x}}{h}$$

$$= \lim_{h \to 0} \frac{(1+x)(1-(x+h)) - (1-x)(1+(x+h))}{h(1+(x+h))(1+x)}$$

$$= \lim_{h \to 0} \frac{-2h}{h(1+(x+h))(1+x)}$$

$$= \lim_{h \to 0} \frac{-2}{(1+(x+h))(1+x)} = \frac{-2}{(1+x)^2}$$

Alternate Notation



Newton Notation

$$\Rightarrow f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Liebniz Notation
(Not really a ratio of two numbers)

$$\frac{dy}{dx} = \text{Derivative as a function}$$

$$\left.\frac{dy}{dx}\right|_{x=a} = \text{Derivative evaluated at } x = a = f'(a)$$

Remark

We say f is <u>differentiable</u> on an open interval I if f differentiable at every a in I.

For example $f(x) = x^3$ is differentiable on $(-\infty, \infty)$

Fact:

f differentiable at $x = a$ $\Rightarrow$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists} \Rightarrow \lim_{h \to 0} f(a+h) - f(a) = 0$$

$$\Rightarrow \lim_{h \to 0} f(a+h) = f(a) \Rightarrow f \text{ continuous at } x = a$$

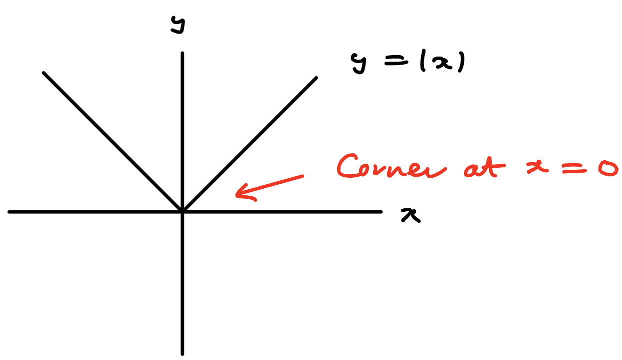
Non-example 1/ $f(x) = |x|$ is not differentiable at $x = 0$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

$$h > 0 \Rightarrow \frac{|h|}{h} = 1 \Rightarrow \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

$$\neq$$

$$h < 0 \Rightarrow \frac{|h|}{h} = -1 \Rightarrow \lim_{h \to 0^-} \frac{|h|}{h} = -1$$



We can repeat process of taking derivative of a function :

Function		First Derivative		Second Derivative		Third Derivative		
$f(x)$	$\longrightarrow$	$f'(x)$	$\longrightarrow$	$f''(x)$	$\longrightarrow$	$f'''(x)$	$\longrightarrow$	$\cdots$
$\parallel$		$\parallel$		$\parallel$		$\parallel$		
y		$\dfrac{dy}{dx}$		$\dfrac{d^2y}{dx^2}$		$\dfrac{d^3y}{dx^3}$		

Example

$s(t)$ = position

$v(t) = s'(t)$ = velocity

$a(t) = v'(t) = s''(t)$ = acceleration.

Future Goal: Develop methods to easily compute derivatives.