The Derivative

Q: How can we measure how quickly $f(x)$ changes as we move x around?

Intuitive Definition

Let f be a function defined on some open interval containing $x = a$.

"f prime at $x=a$"

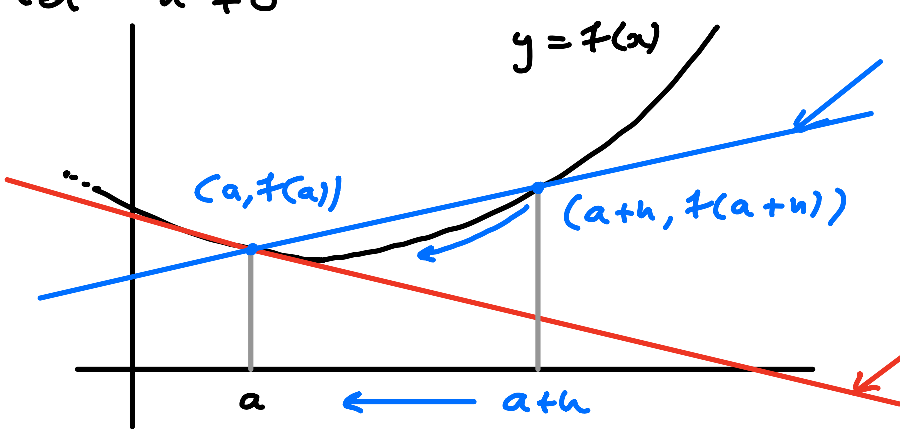
Called the derivative of f at $x=a$

$f'(a)$:= Slope of tangent line to $y=f(x)$ at $x=a$

Alternate Terminology: $f'(a)$ is the instantaneous rate of change of $f(x)$ with respect to x, at $x=a$

How can we calculate $f'(a)$?

Let $h \neq 0$



$$\text{Slope} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

$\text{slope} = f'(a)$

Key Observation :

As $h \to 0$, $\dfrac{f(a+h)-f(a)}{h} \longrightarrow f'(a)$

Precise Definition

We say f is <u>differentiable</u> at $x=a$ if

$\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists. If so we define

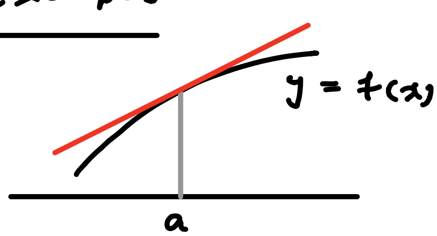
Limit from both sides

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

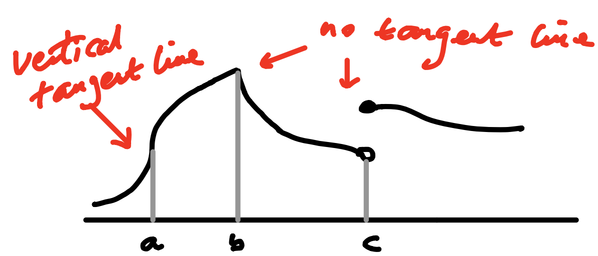
Intuition :

f differentiable at $x=a \iff$ $y=f(x)$ has a <u>non-vertical</u> tangent line at $x=a$

Examples



f differentiable at $x=a$



f non-differentiable at $x=a$, $x=b$, $x=c$

Important Example

$s(t)$ = position on straight line at time t

$s'(a)$ = (instantaneous) velocity at $t=a$

Warning : If domain of f is $[A,B]$

$\Rightarrow$ Can't do limit from both sides at end points

$\Rightarrow$ f non-differentiable at $x = A$ or B.

Examples

1/ $f(x) = x^2$, $a=1$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h}$$

$$= \lim_{h\to 0} \frac{h^2+2h}{h} = \lim_{h\to 0} h+2 = 2$$

2/ $f(x) = \sqrt{x}$, $a = 2$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$$

$$= \lim_{h\to 0} \frac{\overbrace{(\sqrt{2+h} - \sqrt{2})}^{(a-b)}\overbrace{(\sqrt{2+h} + \sqrt{2})}^{(a+b)}}{h(\sqrt{2+h}+\sqrt{2})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{2+h}+\sqrt{2})} = \lim_{h\to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$