The Derivative

Q: How can we measure how quickly \( f(x) \) changes as we move \( x \) around?

**Intuitive Definition**

Let \( f \) be a function defined on some open interval containing \( x = a \).

\( f'(a) \) : Slope of tangent line to \( y = f(x) \) at \( x = a \)

Alternate Terminology: \( f'(a) \) is the instantaneous rate of change of \( f(x) \) with respect to \( x \) at \( x = a \)

How can we calculate \( f'(a) \)?

Let \( h \neq 0 \)

\[
\text{Slope} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}
\]

Slope = \( f'(a) \)
Key Observation:

As \( h \to 0 \), \( \frac{f(a+h) - f(a)}{h} \to f'(a) \)

Precise Definition

We say \( f \) is **differentiable** at \( x = a \) if

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

exists. It so we define

\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

Intuition:

\( f \) differentiable at \( x = a \) \( \iff \)

\( y = f(x) \) has a non-vertical tangent line at \( x = a \)

Examples:

\( f \) differentiable at \( x = a \)

\( f \) non-differentiable at \( x = a, b, c \)

Important Example

\[ y = f(x) \]

\[ a \]

\[ b \]

\[ c \]
\[ s(t) = \text{position on straight line at time } t \]
\[ s'(a) = \text{(instantaneous) velocity at } t = a \]

**Warning:** The domain of \( t \) is \((A, B)\)

\( \Rightarrow \) Can't do limit from both sides at endpoints

\( \Rightarrow \) \( s \) non-differentiable at \( x = A \) or \( B \).

**Examples**

1. \( f(x) = x^2 \), \( a = 1 \)

\[
\begin{align*}
\frac{d}{dx} f(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 2h}{h} = \lim_{h \to 0} h + 2 = 2
\end{align*}
\]

2. \( f(x) = \sqrt{x} \), \( a = 2 \)

\[
\begin{align*}
\frac{d}{dx} f(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \cdot \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{2+h} + \sqrt{2})} = \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} \\
&= \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}
\end{align*}
\]