The Definite Integral

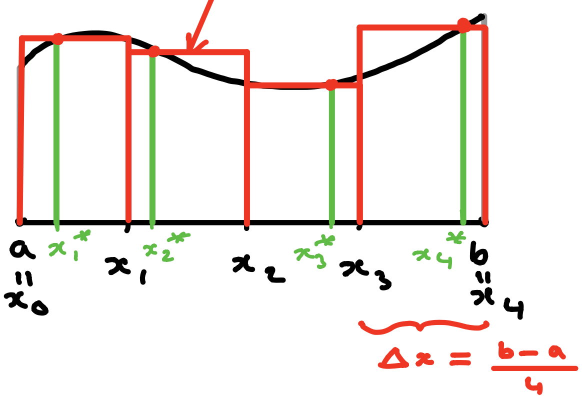
Recall :

f - function on $[a,b]$, $f(x) \geq 0$ for all x in $[a,b]$

$\Rightarrow$ Area bounded by $y = f(x)$ and x-axis between $x = a$ and $x = b$ $= \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} f(x_i^*) \Delta x}_{\text{Riemann Sum}}$

Picture $(n = 4)$

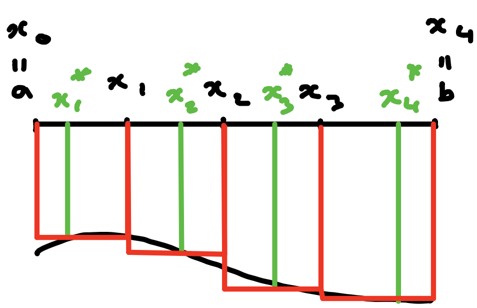
Area $= f(x_2^*)\Delta x$ $\quad$ $y = f(x)$



Observation :

$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$ still makes sense if f is not positive on $[a,b]$.

Assume $f(x) \leq 0$ on $[a,b]$. We have same construction : $(n = 4)$



$\Rightarrow \quad f(x_i^*) = -$ Height of i^{th} rectangle

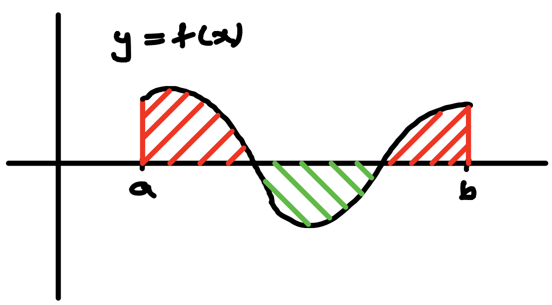
$\Rightarrow \quad f(x_i^*)\Delta x = -$ Area of i^{th} rectangle

$$\Rightarrow \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = -\left(\begin{array}{l}\text{Area bounded by} \\ y = f(x) \text{ and } x\text{-axis is} \\ \text{between } x = a \text{ and} \\ x = b\end{array}\right)$$

<u>More generally</u>:

$\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x =$ <u>Net</u> Area enclosed by $y = f(x)$ and x-axis between $x = a$ and $x = b$

<u>Picture</u>



<u>Net</u> Area = Area ( ) − Area ( )

Basically means concept of Net area is reasonable

<u>Fundamental Definition</u>

We say a function f on $[a,b]$ is <u>integrable</u> if $\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is independent of sample point choices. In this case, we denote this number by

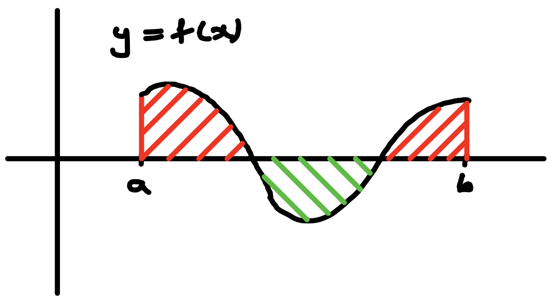
just notation

"one from many"

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Called the <u>definite integral</u> of f on $[a,b]$.

Picture :



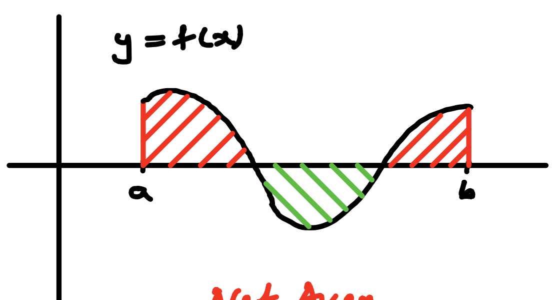
$$\int_a^b f(x)\,dx = \text{Area}(///) - \text{Area}(\backslash\backslash\backslash)$$

could be negative

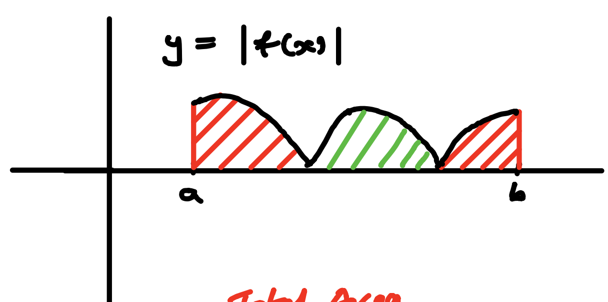
Fact: f cts on $[a,b] \Rightarrow f$ integrable on $[a,b]$

Important : This is not the same as the total area enclosed by $y = f(x)$ and x-axis between a and b

$\text{Area}(///) + \text{Area}(\backslash\backslash\backslash)$.



$\Rightarrow$



Net Area

Total Area

$$\int_a^b f(x)\,dx = \text{Area}(///) - \text{Area}(\backslash\backslash\backslash) \Rightarrow \int_a^b |f(x)|\,dx = \text{Area}(///) + \text{Area}(\backslash\backslash\backslash)$$

Example : Express $\lim_{n \to \infty} \sum_{i=1}^{n} \left(-1 + i\frac{3}{n}\right)^3 \frac{3}{n}$

as a definite integral.

Need to find $\underline{a, b, f(x), x_i^*}$ such that

$$\left(-1 + i\frac{3}{n}\right)^3 \frac{3}{n} = f(x_i^*)\,\Delta x$$

1/ Find appropriate Δx

Recall $\Delta x = \frac{b-a}{n}$, hence choose $\Delta x = \frac{3}{n}$

$\Rightarrow b - a = 3$

Always try $x_i^* = x_i$ first

2/ Choose appropriate a and b.

Recall $x_i = a + i\Delta x$ and we can have

$x_i^* = x_i$, hence choose $a = -1$ and $b = 2$

$\Rightarrow x_i^* = -1 + i\frac{3}{n}$

3/ Choose appropriate f

Need f such that

$f(-1 + i\frac{3}{n})\frac{3}{n} = (-1 + i\frac{3}{n})^3 \frac{3}{n}$, hence choose

$f(x) = x^3$

Conclusion : $\sum_{i=1}^{n} (-1 + i\frac{3}{n})^3 \frac{3}{n} = \int_{-1}^{2} x^3 \, dx$

Properties of Definite Integrals

1/ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ $\quad \frac{a-b}{n} = -\frac{b-a}{n}$

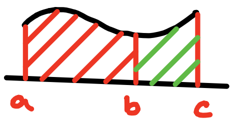
2/ $\int_a^a f(x)\,dx = 0$ $\quad \frac{a-a}{n} = 0$

Sum Law for Limits

3/ $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4/ $\displaystyle\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$ ← Constant Multiple Law for limits

5/ $a < b < c \Rightarrow \displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



6/ $f(x) \le g(x)$ on $[a,b]$ $\Rightarrow \displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$