Curve Sketching

Aim : Sketch graph of $y=f(x)$

↓

Important to really understand core functions

What is domain of f ?

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Rotational symmetry rotating by π about $(0,0)$

Reflective symmetry in y-axis

Is f odd or even ?

↓

Are there vertical asymptotes ?

$x=a$ vertical asymptote $\Leftrightarrow \lim_{x \to a^+ / a^- / a} f(x) = \pm\infty$

↓

What is the behavior at ∞ and $-\infty$?

$\lim_{x \to \pm\infty} f(x) = \infty / -\infty \Rightarrow$ $y=f(x)$ grows positively / negatively without bound as $x \to \pm\infty$.

$\lim_{x \to \pm\infty} f(x) = L \Rightarrow$ $y=L$ is a horizontal asymptote

$\lim_{x \to \pm\infty} f(x) - (mx+b) = 0 \Rightarrow$ $y=mx+b$ a <u>slant</u> asymptot

We don't generally do this one

Sign Analysis on f

Gives x-intercepts and tells whether $f(x)$ + or −.

Example

$f(c) = 0$ or f discontinuus at c

$y = f(x)$ above x-axis

$y = f(x)$ below x-axis

\+ c − $f(x)$

$f(x) > 0$ $f(x) < 0$

Sign Analysis on f'

Tells us where f increasing / decreasing and gives local max/min

Example

f continuous at d $\Rightarrow$ $f(d)$ local max

f increasing

f decreasing

\+ d − $f'(x)$

$f'(x) > 0$ $f'(x) < 0$

Sign Analysis on f'

Tells us concavity and inflection points.

Example

f continuous at e $\Rightarrow$ $(e, f(e))$ Inflection

f concave down

f concave up

− e + $f''(x)$

$f''(x) < 0$ $f''(x) > 0$

Start by plotting x/y-intercepts, local max/min and asymptotes.

Put it all together to sketch $y = f(x)$

Remark

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = m \neq 0 \quad \underline{\text{and}} \quad \lim_{x \to \pm\infty} (f(x) - mx) = b$$

$\Rightarrow \quad y = mx + b$ is slant asymptote.

Example Sketch $y = \dfrac{x^2}{x+1} = f(x)$

Domain: $(-\infty, -1) \cup (-1, \infty)$

Odd / Even: Neither

Vertical Asymptotes:

$$\lim_{x \to -1^+} x^2 = 1 > 0$$

$$\lim_{x \to -1^+} x+1 = 0^+$$

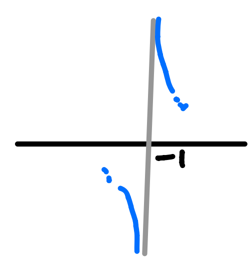
$$\Rightarrow \lim_{x \to -1^+} \frac{x^2}{x+1} = \infty$$

$$\lim_{x \to -1^-} x^2 = 1 > 0$$

$$\lim_{x \to -1^-} x+1 = 0^-$$

$$\Rightarrow \lim_{x \to -1^+} \frac{x^2}{x+1} = -\infty$$

$\Rightarrow \quad x = -1$ is vertical asymptote.



Behavior at $\pm\infty$:

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^2}{x^2 + x} = 1$$

$$\lim_{x \to \pm\infty} f(x) - x = \lim_{x \to \pm\infty} \frac{x^2}{x+1} - x = \lim_{x \to \pm\infty} \frac{x^2 - x^2 - x}{x+1}$$

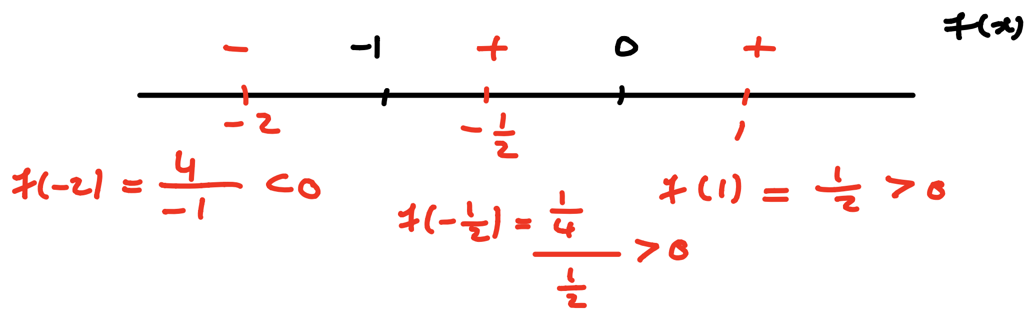
$$= \lim_{x \to \pm\infty} \frac{-x}{x+1} = -1$$

$\Rightarrow \quad y = x - 1$ is a slant asymptote

Sign Analysis on f

A/ $f(x) = 0 \iff \dfrac{x^2}{x+1} = 0 \iff x = 0$

B/ f discontinuous at $x \iff x = -1$

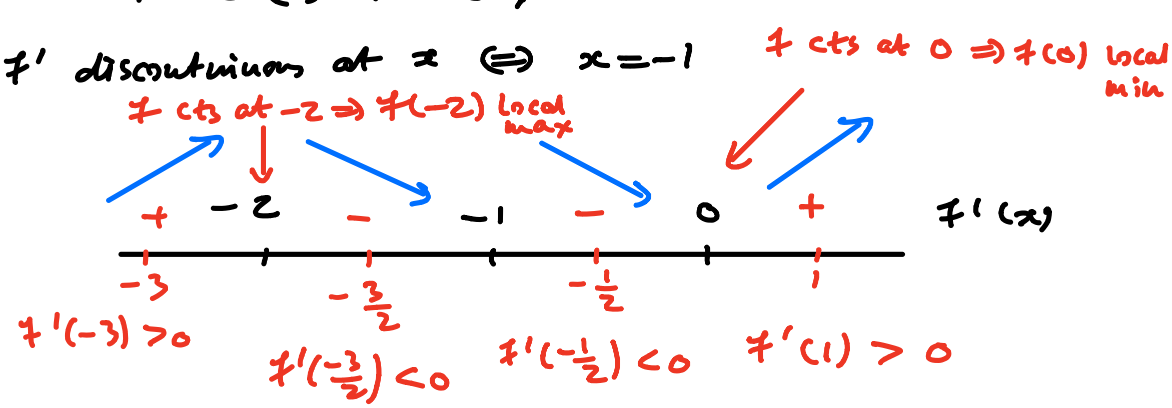


Sign Analysis of f'

$$f'(x) = \frac{2x(x+1) - 1 \cdot x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

A/ $f'(x) = 0 \iff x = 0, -2$

B/ f' discontinuous at $x \iff x = -1$



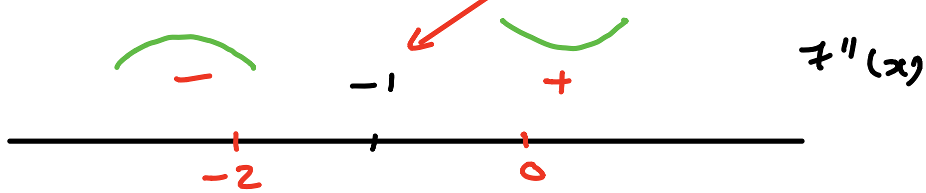
Sign Analysis of f''

$$f''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)\,x(x+2)}{(x+1)^4} = \frac{2}{(x+1)^3}$$

A/ $\frac{2}{(x+1)^3} \neq 0$

B/ f'' discontinuous at $x = -1$

f not cts at -1
$\Rightarrow$ Not inflection



$f''(-2) = -2 < 0$ $\qquad$ $f''(0) = 2 > 0$