Continuity

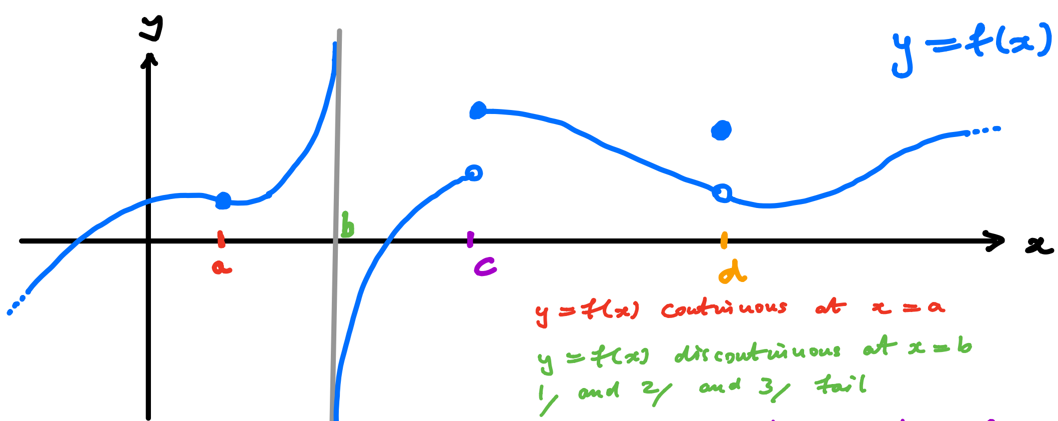
Definition A function f is continuous at $x = a$ if

1/ $f(a)$ is defined, ie a is in domain of f

2/ $\lim\limits_{x \to a} f(x)$ exists

3/ $\lim\limits_{x \to a} f(x) = f(a)$

If any of these three conditions fail we say f is discontinuous at $x = a$



$y = f(x)$ continuous at $x = a$

$y = f(x)$ discontinuous at $x = b$
1/ and 2/ and 3/ fail

$y = f(x)$ discontinuous at $x = c$
2/ and 3/ fail

$y = f(x)$ discontinuous at $x = d$
3/ fails

Intuition : $y = f(x)$ is continuous at $x = a$ if we can draw graph through $(a, f(a))$ without lifting pen.

Remarks

1/ In the definition we could replace $\lim_{x \to a} f(x)$

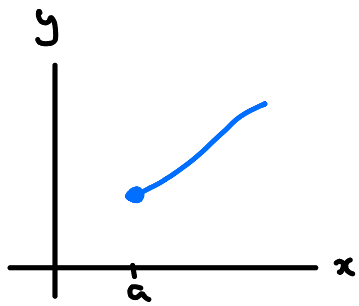
$\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$.

(right)

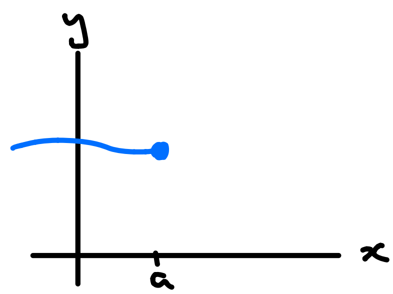
$\lim_{x \to a^+} f(x) = f(a) \iff$ f is continuous from above at $x = a$

(left)

$\lim_{x \to a^-} f(x) = f(a) \iff$ f is continuous from below at $x = a$



Continuous from above



Continuous from below

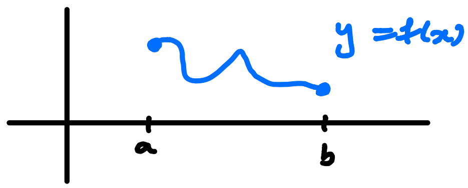
2/ f is said to be continuous on an interval I if it is continuous at every a in I.

From above / below at possible endpoints

3/ f is said to be continuous if it is continuous at every a in its domain.

From above / below at possible endpoints

Example domain = $[a,b]$ (closed interval)



$\Rightarrow f$ continuous

Polynomials, Rational functions, Power Functions, Trig functions, Inverse Trig functions, Exponential functions and Log functions are continuous

Facts

1/ f, g continuous at $x = a$ $\Rightarrow$ $f+g,\ fg,\ \overset{\text{constant}}{c}f,\ \dfrac{f}{g}$ continuous at $x = a$

(For quotient need $g(a) \neq 0$)

2/ $\lim\limits_{x \to a} g(x) = b$

and

f continuous at b $\Rightarrow$ $\lim\limits_{x \to a} f(g(x)) = f(b)$

1/, 2/ $\Rightarrow$ The sum / product / quotient / composition of continuous functions is again continuous (although the domain may change).

Examples

1/ $\lim\limits_{x \to 3} \sin\left(\dfrac{x^2+1}{x+2}\right) = ?$

$g(x) = \dfrac{x^2+1}{x+2}$, $f(x) = \sin(x)$ continuous

$\Rightarrow \sin\left(\dfrac{x^2+1}{x+2}\right)$ continuous.

3 in domain $\Rightarrow \lim\limits_{x \to 3} \sin\left(\dfrac{x^2+1}{x+2}\right) = \sin(2)$

2/ $\lim\limits_{x \to 0} \arctan\left(e^{(x^2 \sin(1/x))}\right) = ?$

Recall : By Squeeze Theorem $\lim\limits_{x \to 0} x^2 \sin(1/x) = 0$

e^x continuous at $x = 0$

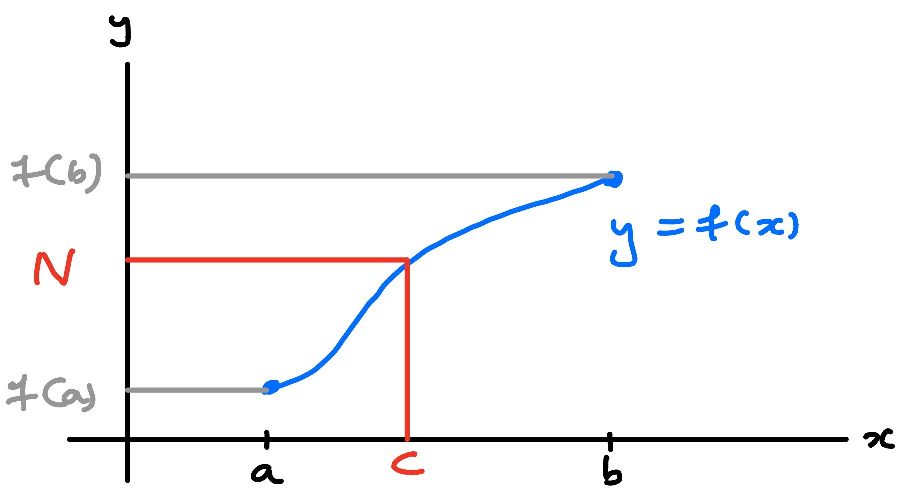
$\Rightarrow \lim\limits_{x \to 0} e^{(x^2 \sin(1/x))} = e^0 = 1$

$\arctan(x)$ continuous at $x = 1$

$\Rightarrow \lim\limits_{x \to 0} \arctan\left(e^{(x^2 \sin(1/x))}\right) = \arctan(1) = \dfrac{\pi}{4}$

Intermediate Value Theorem

Let f be continuous on $[a,b]$ and $f(a) \neq f(b)$. If N is between $f(a)$ and $f(b)$ then there exists c in $[a,b]$ such that $f(c) = N$.



Example

Show that for $f(x) = \dfrac{x^3 + x + 1}{\sin(x) + 2}$ there exists c in $\mathbb{R}$ such that $f(c) = 0$.

Observe that domain is all $\mathbb{R}$ as $\sin(x) + 2 \geq 1$ for all x in $\mathbb{R}$.

Observe $f(-2) = \dfrac{-4}{\sin(-2)+2} < 0$

$$f(1) = \frac{3}{\sin(1)+2} > 0$$

I.V.T.

f continuous on $[-2,1]$ $\Rightarrow$ there exist c in $[-2,1]$ such that $f(c) = 0$.