Continuity

**Definition**

A function \( f \) is continuous at \( x = a \) if

1. \( f(a) \) is defined, i.e., \( a \) is in the domain of \( f \)
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \)

If any of these three conditions fail, we say \( f \) is discontinuous at \( x = a \).

**Intuition:** \( y = f(x) \) is continuous at \( x = a \) if we can draw the graph through \((a, f(a))\) without lifting pen.
Remarks

1. In the definition we could replace \( \lim f(x) \) as \( \lim f(x) \) or \( \lim f(x) \).

\[
\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x). \quad (\text{right})
\]

\[
\lim_{x \to a^+} f(x) = f(a) \quad \Leftrightarrow \quad f \text{ is continuous from above at } x = a \quad (\text{left})
\]

\[
\lim_{x \to a^-} f(x) = f(a) \quad \Leftrightarrow \quad f \text{ is continuous from below at } x = a
\]

\[
\begin{array}{c}
\text{Continuous from above} \\
a \\
\text{Continuous from below}
\end{array}
\]

2. \( f \) is said to be continuous on an interval \( I \) if it is continuous at every \( a \) in \( I \).

3. \( f \) is said to be continuous if it is continuous at every \( a \) in its domain.

\[
\begin{array}{c}
\text{From above/below} \\
at \text{ possible endpoints}
\end{array}
\]

\[
\begin{array}{c}
\text{From above/below} \\
at \text{ possible endpoints}
\end{array}
\]
Example domain = \([a,b]\) (closed interval)

\[ y = f(x) \quad \Rightarrow \quad f \text{ continuous} \]

Polynomials, Rational functions, Power functions, Trig functions, Inverse Trig functions, Exponential functions and Log functions are continuous.

Facts

1. If \( f, g \) continuous at \( x = a \) \( \Rightarrow \)

\( f + g, f g, c f, \frac{f}{g} \) continuous at \( x = a \)  
(For quotient need) \( g(a) \neq 0 \)

\[
\lim_{{x \to a}} g(x) = b \\
\text{and} \\
\lim_{{x \to a}} f(g(x)) = f(b)
\]

The sum / product / quotient / composition

\( 1, 2, 3 \Rightarrow \) of continuous functions is again continuous (although the domain may change).
**Examples**

1. \( \lim_{x \to 3} \sin \left( \frac{x^2+1}{x+2} \right) = ? \)

   \[ g(x) = \frac{x^2+1}{x+2}, \quad f(x) = \sin(x) \quad \text{continuous} \]

   \[ \Rightarrow \sin \left( \frac{x^2+1}{x+2} \right) \quad \text{continuous} \]

   \[ \text{3 in domain} \Rightarrow \lim_{x \to 3} \sin \left( \frac{x^2+1}{x+2} \right) = \sin(3) \]

2. \( \lim_{x \to 0} \arctan \left( e^{x^2 \sin(1/x)} \right) = ? \)

   Recall: By Squeeze Theorem \( \lim_{x \to 0} x^2 \sin(1/x) = 0 \)

   \( e^x \quad \text{continuous at } x = 0 \)

   \[ \Rightarrow \lim_{x \to 0} e^{x^2 \sin(1/x)} = e^0 = 1 \]

   \( \arctan(x) \quad \text{continuous at } x = 1 \)

   \[ \Rightarrow \lim_{x \to 0} \arctan \left( e^{x^2 \sin(1/x)} \right) = \arctan(1) = \frac{\pi}{4} \]
**Intermediate Value Theorem**

Let \( f \) be continuous on \([a, b]\) and \( f(a) \neq f(b) \). If \( N \) is between \( f(a) \) and \( f(b) \) then there exists \( c \) in \([a, b]\) such that \( f(c) = N \).

**Example**

Show that for \( f(x) = \frac{3}{x^3 + x + 1} \) there exists \( c \) in \( \mathbb{R} \) such that \( f(c) = 0 \).

Observe that domain is all \( \mathbb{R} \) as \( \sin(x) + 2 \geq 1 \) for all \( x \) in \( \mathbb{R} \).
Observe \( f(-2) = \frac{-9}{\sin(-2)+2} < 0 \)
\( f(1) = \frac{3}{\sin(1)+2} > 0 \) \( \text{I.V.T.} \)

\( f \) continuous on \([-2, 1]\) \( \Rightarrow \) there exist \( c \) in \([-2, 1]\) such that \( f(c) = 0 \).