Computing Limits

Core Examples

If f is a polynomial / rational function / power function / trig function / inverse trig function / exponential function / logarithm function such that $f(x)$ is defined on open interval containing $x=a$, then

$$\lim_{x \to a} f(x) = f(a)$$

Examples

$$\lim_{x \to 1} e^x = e^1 = e, \quad \lim_{x \to 2} \frac{x-1}{x^2+2} = \frac{2-1}{2^2+2} = \frac{1}{6}$$

Remarks For arcsin, arccos, power functions same holds at endpoints. For example

$$\lim_{x \to -1^+} \arccos(x) = \pi, \quad \lim_{x \to 0^+} \sqrt{x} = 0$$

Warning: This is absolutely untrue in general. $f(a)$ may not even be defined

Fact If $f(x) = g(x)$ for $x \neq a \Rightarrow$

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a} g(x) = L$$

Example

assuming $x \neq 1$

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1 = g(x)$$

polynomial

$$\Rightarrow \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} x+1 = 1+1 = 2$$

Limit Laws

(All assuming $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ exist)

Sum Law: $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} (f(x)) + \lim_{x \to a} (g(x))$

Constant Multiple Law: $\lim_{x \to a} (c f(x)) = c \cdot \lim_{x \to a} f(x)$

Product Law: $\lim_{x \to a} (f(x) g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

Power Law: If x^r defined on open interval containing $\lim_{x \to a} f(x)$

$$\lim_{x \to a} (f(x))^r = \left(\lim_{x \to a} f(x)\right)^r$$

Quotient Law:

$$\lim_{x \to a} g(x) \neq 0 \Rightarrow \lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

Example

$$\lim_{x \to 1} \left(\frac{\sqrt{x^2+1}}{x+3}\right) = ?$$

$$\lim_{x \to 1} x+3 = 1+3 = 4 \neq 0$$

$\sqrt{x}$ defined near 2

$$\lim_{x \to 1} x^2+1 = 1^2 + 1 = 2$$

$$\Rightarrow \lim_{x \to 1} \frac{\sqrt{x^2+1}}{x+3} = \frac{\sqrt{\lim_{x\to 1} x^2+1}}{\lim_{x \to 1} x+3} = \frac{\sqrt{2}}{4}$$

Fact:

$$\lim_{x\to a} f(x) = L \neq 0 \;,\; \lim_{x\to a} g(x) = 0 \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} \text{ DNE}$$

approaches 0 positively

$$\lim_{x\to a} f(x) = L > 0 \;,\; \lim_{x\to a} g(x) = 0^+ \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \infty$$

approaches 0 negatively

$$\lim_{x\to a} f(x) = L > 0 \;,\; \lim_{x\to a} g(x) = 0^- \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = -\infty$$

Remark: All above hold for $x \to a^{-1} / a^+$

Example $f(x) = e^x + 2$, $g(x) = x^2$, $a = 0$

$$\lim_{x\to 0} e^x + 2 = e^0 + 2 = 3$$

$$\lim_{x\to 0} x^2 = 0^+$$

square always non-negative

$$\Rightarrow \lim_{x\to 0} \frac{e^x+2}{x^2} = \infty$$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then anything could happen to $\lim_{x\to a} \frac{f(x)}{g(x)}$.

Examples

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

$$\lim_{x \to 0} \frac{2x}{x} = \lim_{x \to 0} 2 = 2$$

$$\lim_{x \to 0} \frac{x}{x^3} = \lim_{x \to 0} \frac{1}{x^2} = \infty$$

<u>Important Strategy</u> :

If $\lim_{x \to a} f(x) = 0$ <u>and</u> $\lim_{x \to a} g(x) = 0$,

try to simplify $\frac{f(x)}{g(x)}$ before applying Limit Laws.

<u>Example</u> $\lim_{x \to 0} \frac{\sqrt{x^2+1} - 1}{x^2} = ?$

$$\lim_{x \to 0} \left(\sqrt{x^2+1} - 1\right) = \sqrt{0^2+1} - 1 = 0$$

$$\lim_{x \to 0} x^2 = 0^2 = 0$$

$$\frac{\sqrt{x^2+1} - 1}{x^2} = \frac{\left(\sqrt{x^2+1} - 1\right)\left(\sqrt{x^2+1} + 1\right)}{x^2\left(\sqrt{x^2+1} + 1\right)}$$

$$= \frac{(x^2+1) - 1^2}{x^2\left(\sqrt{x^2+1} + 1\right)} \overset{x \neq 0}{=} \frac{1}{\left(\sqrt{x^2+1} + 1\right)}$$

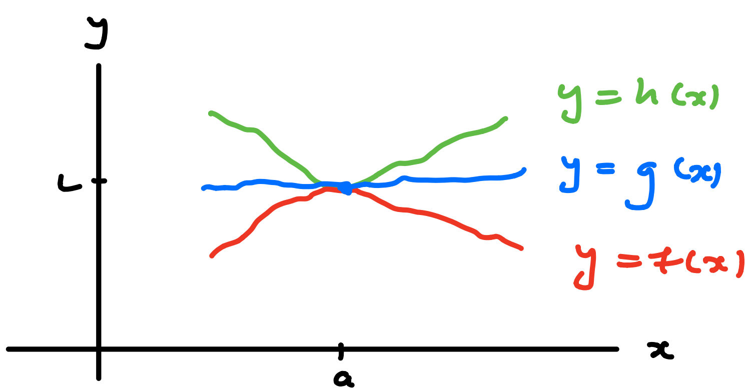
$$\Rightarrow \lim_{x \to 0} \frac{\sqrt{x^2+1} - 1}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+1} + 1} = \frac{1}{2}$$

Squeeze Theorem

If $\quad f(x) \le g(x) \le h(x)$ near $x = a$ then

$$\left.\begin{array}{l} \lim\limits_{x\to a} f(x) = L \\ \underline{\text{and}} \\ \lim\limits_{x\to a} h(x) = L \end{array}\right. \Rightarrow \lim_{x\to a} g(x) = L$$

Picture



Example

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = ?$$

$\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$ DNE $\Rightarrow$ Cannot use product law

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \quad \Rightarrow \quad -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0 \Rightarrow \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$