The Chain Rule

Recall: Let \( y = t(x) \).

\[
\frac{dy}{dx} \quad \text{Instantaneous rate of change of } y \text{ with respect to } x.
\]

Q: What happens when \( y \) can be expressed in terms of a different independent variable?

Motivating Example:

A leaking oil well is spreading a circular film over a water surface. At time \( t \) after the start of the leak the radius is \( 4t \).

Let \( A = \text{surface area of spill} \), \( r = \text{radius of spill} \) and \( t = \text{time after start of leak} \).

A can be a function in \( r \) or \( t \). Is there a relationship between \( \frac{dA}{dr} \) and \( \frac{dA}{dt} \)?

\[
A(r) = \pi r^2
\]

\[
A(t) = \pi (r(t))^2 = \pi (4t)^2 = 16\pi t^2
\]

\[
\Rightarrow \frac{dA}{dr} = 2\pi r \quad \text{and} \quad \frac{dA}{dt} = 32\pi t
\]
Observations

\[
\frac{dA}{dr} = 2\pi r = 2\pi (4t) = 8\pi t \\
\frac{dr}{dt} = \frac{d}{dt} (4t) = 4 \\
\Rightarrow \frac{dA}{dt} = 32\pi t = 8\pi t \cdot 4 = \frac{dA}{dr} \cdot \frac{dr}{dt}
\]

Conclusion:

Rate of Change of A with respect to t = Rate of Change of A with respect to r \times Rate of change of r with respect to t

Chain Rule (Form 1)

\[y = f(u) \quad \text{\small y is a function in u.} \]
\[u = g(x) \quad \text{\small u is a function in t} \]
\[\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{\small Clever Notation.} \]
\[\quad \text{\small Not really cancellation of fractions.} \]

Example

\[y = \sqrt{u} \quad \Rightarrow \frac{dy}{dx} = ? \]
\[u = 2^x \tan(x) \]
If \( \frac{dy}{du} = \frac{d}{du} \left( u^{\frac{1}{2}} \right) = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} \)  

P. R.  
\[ \frac{du}{dx} = \ln(2) 2^{x} \tan(x) + 2^{x} \sec^{2}(x) \]

\[ \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

Need to switch to \( x \)
\[ = \frac{1}{2\sqrt{u}} \cdot \left( \ln(2) 2^{x} \tan(x) + 2^{x} \sec^{2}(x) \right) \]
\[ = \left( \ln(2) 2^{x} \tan(x) + 2^{x} \sec^{2}(x) \right) \frac{1}{2 \sqrt{2^{x} \tan(x)}} \]

\[ y = \sec(u) \Rightarrow \frac{dy}{dx} = ? \]

\[ u = \cos(x) \]

\[ \frac{dy}{du} = \tan(u) \sec(u) \]

\[ \frac{du}{dx} = -\sin(x) \]

\[ \Rightarrow \frac{dy}{dx} = \tan(u) \sec(u) \cdot (-\sin(x)) \]
\[ = \tan(\cos(x)) \sec(\cos(x)) \cdot (-\sin(x)) \]

Prime Notation:
\[ y = f(u) \Rightarrow \frac{dy}{du} = f'(u) = f'(g(x)) \]
\[ u = g(x) \Rightarrow \frac{dy}{du} = f'(u) = f'(g(x)) \]
\[ \frac{du}{dx} = g'(x) \]
Conclusion: The Chain Rule tells us how to differentiate compositions of functions.

Remark:

¢ You can use either form. The advantage of Liebniz notation is that it explicitly tells us the relevant independent variable. For example is $y'$ equal to $\frac{dy}{dx}$ or $\frac{dy}{dt}$?

¢ We can extend the Chain Rule to compositions of more functions. For example:

\[ y = f(u) \]
\[ u = g(x) \Rightarrow \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt} \]
\[ x = h(t) \]

\[ \frac{d}{dt} (f(g(h(t)))) = f'(g(h(t))) \cdot g'(h(t)) \cdot h'(t) \]
Example

\[
\frac{dy}{dt} (2 \sin(t^2)) = ?
\]

\[
y = 2^u
\]
\[
u = \sin(x) \Rightarrow \frac{du}{dt} = \frac{du}{dz} \cdot \frac{dz}{dt} = 1 \cdot 2^u \cdot \cos(x) \cdot z \cdot t
\]
\[
z = t^2 = \frac{dz}{dt} = 1 \cdot 2^u \cdot \sin(t^2) \cdot \cos(t^2) \cdot 2t
\]

Useful Standard Examples:

\[
1. \quad \frac{d}{dx} \left( \pm (mx + b) \right) = \pm (mx + b)' \cdot m
\]

\[
2. \quad \frac{d}{dx} \left( b g(x)^m \right) = \frac{d}{du} \left( b \right) \cdot \frac{d}{du} \left( g(x)^m \right) \cdot g'(x)
\]

\[
3. \quad \frac{d}{dx} \left( g(x)^r \right) = r \cdot g(x)^{r-1} \cdot g'(x)
\]

Product Rule + Chain Rule \Rightarrow Quotient Rule

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{d}{dx} \left( f(x) \cdot (g(x))^{-1} \right)
\]

\[
= \frac{d}{dx} \left( f(x) \right) \cdot (g(x))^{-1} + f(x) \frac{d}{dx} \left( (g(x))^{-1} \right)
\]

\[
= \frac{f'(x)}{g(x)} + f(x) \cdot \frac{(-1)}{(g(x))^2} \cdot g'(x)
\]
\[ = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \]