The Chain Rule

Recall : Let $y = f(x)$. Independent variable

$\frac{dy}{dx}$ = Instantaneous rate of change of y with respect to x.

Q: What happens when y can be expressed in terms of a different independent variable?

Motivating Example :

A leaking oil well is spreading a circular film over a water surface. At time t after the start of the leak the radius is $4t$.

Let A = surface area of spill, r = radius of spill and t = time after start of leak?

A can be a function in r or t. Is there a relationship between $\frac{dA}{dr}$ and $\frac{dA}{dt}$?

$$A(r) = \pi r^2$$

$$A(t) = \pi (r(t))^2 = \pi (4t)^2 = 16\pi t^2$$

$$\Rightarrow \frac{dA}{dr} = 2\pi r \quad \text{and} \quad \frac{dA}{dt} = 32\pi t$$

Observations

$$\frac{dA}{dr} = 2\pi r = 2\pi \cdot (4t) = 8\pi t$$

$$\frac{dr}{dt} = \frac{d}{dt}(4t) = 4$$

$$\Rightarrow \frac{dA}{dt} = 32\pi t = 8\pi t \cdot 4 = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

Conclusion :

Rate of Change of A with respect to t = Rate of Change of A with respect to r × Rate of change of r with respect to t

Chain Rule (Form 1)

$y = f(u)$ ← y is a function in u.

$u = g(x)$ ← u is a function in t

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ ← Clever Notation. Not really cancellation of fractions.

Example

1/ $y = \sqrt{u}$

$u = 2^x \tan(x)$

$\Rightarrow \frac{dy}{dx} = ?$

$$\frac{dy}{du} = \frac{d}{du}\left(u^{\frac{1}{2}}\right) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

P.R.

$$\frac{du}{dx} = \ln(2)\,2^x \tan(x) + 2^x \sec^2(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot \left(\ln(2)\,2^x \tan(x) + 2^x \sec^2(x)\right)$$

Need to switch to x

$$= \frac{\left(\ln(2)\,2^x \tan(x) + 2^x \sec^2(x)\right)}{2\sqrt{2^x \tan(x)}}$$

2/ $y = \sec(u)$, $u = \cos(x)$ $\Rightarrow \frac{dy}{dx} = ?$

$$\frac{dy}{du} = \tan(u)\sec(u)$$

$$\frac{du}{dx} = -\sin(x)$$

$$\Rightarrow \frac{dy}{dx} = \tan(u)\sec(u) \cdot (-\sin(x))$$

$$= \tan(\cos(x))\sec(\cos(x)) \cdot (-\sin(x))$$

Prime Notation :

$y = f(g(x))$ composition

$y = f(u)$, $u = g(x)$ $\Rightarrow$

$$\frac{dy}{du} = f'(u) = f'(g(x))$$

$$\frac{du}{dx} = g'(x)$$

$\Rightarrow$

Chain Rule (Form 2)

$$\underset{\frac{dy}{dx}}{\nearrow}\ \frac{d}{dx}\left(f(g(x))\right) = \underset{\frac{dy}{du}}{f'(g(x))\nearrow}\ \underset{\frac{du}{dx}}{g'(x)\nearrow}$$

Conclusion: The Chain Rule tells us how to differentiate compositions of functions

Remark

1/ You can use either form. The advantage of Liebniz notation is that it explicitly tells us the relevant independent variable. For example is y' equal to $\frac{dy}{dx}$ or $\frac{dy}{dt}$?

2/ We can extend the Chain Rule to compositions of more functions. For example

$$\left.\begin{array}{l} y = f(u) \\ u = g(x) \\ x = h(t) \end{array}\right. \Rightarrow \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}$$

$$\Updownarrow$$

$$\frac{d}{dt}\left(f(g(h(t)))\right) = f'(g(h(t))) \cdot g'(h(t)) \cdot h'(t)$$

Example

1/ $$\frac{d}{dt}\left(2^{\sin(t^2)}\right) = ?$$

$$y = 2^u$$
$$u = \sin(x) \quad \Rightarrow \quad \frac{dy}{dt} = \ln(2)2^u \cdot \cos(x) \cdot 2t$$
$$x = t^2 \qquad\qquad = \ln(2)2^{\sin(t^2)} \cdot \cos(t^2) \cdot 2t$$

Useful Standard Examples :

1/ $$\frac{d}{dx}(f(mx+b)) = f'(mx+b) \cdot m$$

2/ $$\frac{d}{dx}\left(b^{g(x)}\right) = \ln(b)\, b^{g(x)} \cdot g'(x)$$

3/ $$\frac{d}{dx}\left((g(x))^r\right) = r(g(x))^{r-1} \cdot g'(x)$$

Product Rule + Chain Rule $\Rightarrow$ Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) \cdot (g(x))^{-1}\right)$$

$$= \frac{d}{dx}(f(x)) \cdot (g(x))^{-1} + f(x)\frac{d}{dx}\left((g(x))^{-1}\right)$$

$$= \frac{f'(x)}{g(x)} + f(x) \cdot \frac{(-1)}{(g(x))^2}\, g'(x)$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$