Areas Between Curves

Motivating Example:

Suppose there are two populations A and B.

$P_A(t)$ = Size of population A at time t.

$P_B(t)$ = Size of population B at time t.

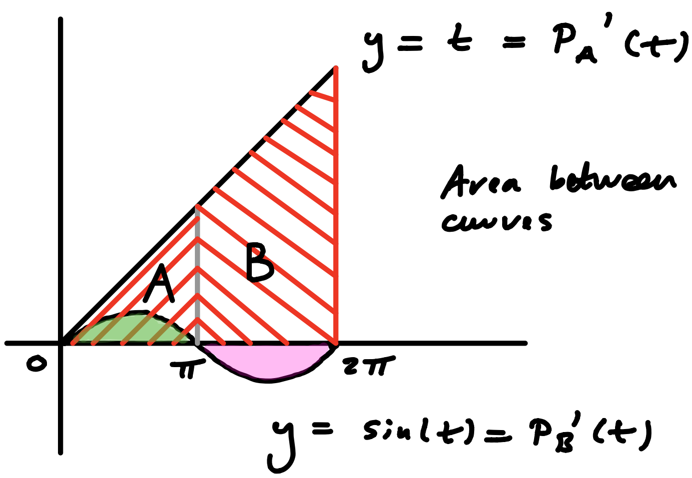
Assume $P_A'(t) = t$ and $P_B'(t) = \sin(t)$ ← Growth rate of populations

and $P_A(0) = P_B(0)$.

1/ Interpret the area between the curves

$y = P_A'(t)$ and $y = P_B'(t)$ between $t = 0$ and $t = 2\pi$

2/ Calculate this area.



Area between curves = Area (A) + Area (B)

$$\text{Area}(A) = \text{Area}(\;) - \text{Area}(\;)$$

$$= \int_0^{\pi} P_A'(t)\,dt - \int_0^{\pi} P_B'(t)\,dt = \int_0^{\pi} \left(P_A'(t) - P_B'(t)\right)dt$$

$$\text{Area}(B) = \text{Area}(\;) + \text{Area}(\;)$$

under x-axis so area is counted negatively

$$= \int_{\pi}^{2\pi} P_A'(t)\,dt - \int_{\pi}^{2\pi} P_B'(t)\,dt = \int_{\pi}^{2\pi} \left(P_A'(t) - P_B'(t)\right)dt$$

$\Rightarrow$

$$\text{Area between curves} = \int_0^{2\pi} P_A'(t) - P_B'(t)\;dt$$

$P_A'(t) \geq P_B'(t)$ on $[0, 2\pi]$

FTC $\longrightarrow$

$$= P_A(t) - P_B(t)\Big|_0^{2\pi}$$

$P_A(0) = P_B(0)$

$$= P_A(2\pi) - P_B(2\pi)$$

Conclusion

Area between curves between $t=0$ and $t=2\pi$ = Difference between sizes of populations A and B at time $t = 2\pi$.

Interpretation

$$= \int_0^{2\pi} t - \sin(t)\,dt$$

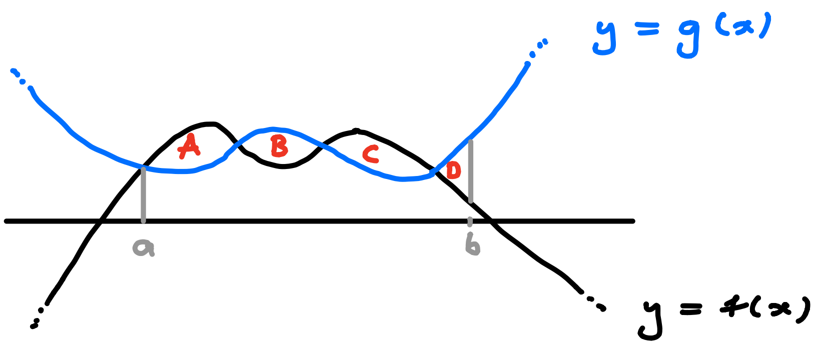
$$= \frac{1}{2}t^2 + \cos(t)\Big|_0^{2\pi}$$

$$= \left(\frac{1}{2}(2\pi)^2 + \cos(2\pi)\right) - \left(0 + \cos(0)\right)$$

$$= 2\pi^2$$

Q: Given f and g, functions on $[a,b]$, what is the total area enclosed by $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$?

Picture



Important Observation :

Total area enclosed = Area (A) + Area (B) + Area (C) + Area (D)

$$\neq$$

$$\int_a^b f(x) - g(x)\,dx = \text{Area}(A) - \text{Area}(B) + \text{Area}(C) - \text{Area}(D)$$

Strategy: Find area of each region individually.

Remark :

Not very useful in practice.

Total area enclosed by $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ = Total area enclosed by $y = f(x) - g(x)$ and x-axis between $x = a$ and $x = b$ $= \displaystyle\int_a^b |f(x) - g(x)|\,dx$

Calculate by doing sign analysis on $f(x) - g(x)$ and calculating $\left|\displaystyle\int_{a_1}^{a_2} f(x) - g(x)\,dx\right|$ on each subinterval

Example

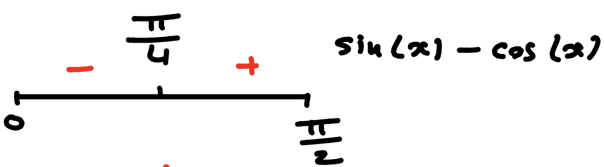
Find total area enclosed by $y = \sin(x)$ and $y = \cos(x)$ between $t = 0$ and $t = \frac{\pi}{2}$.

1/ Do sign analysis on $\sin(x) - \cos(x)$

A/ $\sin(x) - \cos(x) = 0 \iff \dfrac{\sin(x)}{\cos(x)} = 1$

$\iff \tan(x) = 1 \iff x = \dfrac{\pi}{4}$

B/ $\sin(x) - \cos(x)$ cts on $[0, \frac{\pi}{2}]$

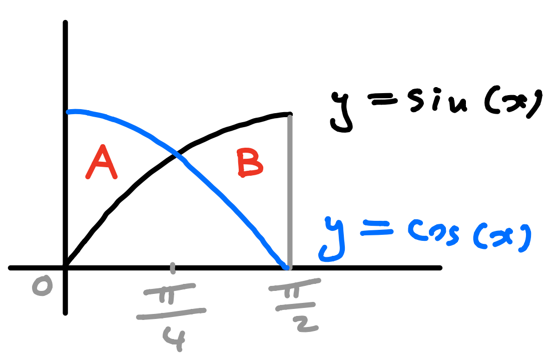


$\sin(0) - \cos(0) < 0$

$\sin(\frac{\pi}{2}) - \cos(\frac{\pi}{2}) > 0$

$\Rightarrow$ There are two regions we must calculate area for.

Picture



$$\text{Area}(A) = -\int_0^{\pi/4} \sin(x) - \cos(x)\, dx$$

$\sin(x) - \cos(x) \leq 0$
on $[0, \frac{\pi}{4}]$

$$= -\left(-\cos(x) - \sin(x)\right)\Big|_0^{\frac{\pi}{4}}$$

$$= -\left(\left(-\cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)\right) - \left(-\cos(0) - \sin(0)\right)\right)$$

$$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

$$\text{Area}(B) = \int_{\frac{\pi}{4}}^{\pi/2} \sin(x) - \cos(x)\, dx$$

$\sin(x) - \cos(x) \geq 0$

on $[\frac{\pi}{4}, \frac{\pi}{2}]$

$$= \left(-\cos(x) - \sin(x)\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \left(-\cos\left(\tfrac{\pi}{2}\right) - \sin\left(\tfrac{\pi}{2}\right)\right) - \left(-\cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)\right)$$

$$= \sqrt{2} - 1$$

$\Rightarrow$ Total area enclosed $= 2(\sqrt{2} - 1)$

<u>Conclusion</u> : To calculate <u>total</u> area enclosed by $y = f(x)$ and $y = g(x)$ between $t = a$ and $t = b$:

1/ Do sign analysis on $f(x) - g(x)$ on $[a, b]$

2/ Calculate $\left|\int_{a_1}^{a_2} f(x) - g(x)\, dx\right|$ on each subinterval to give area of each enclosed region

3. Add these numbers.

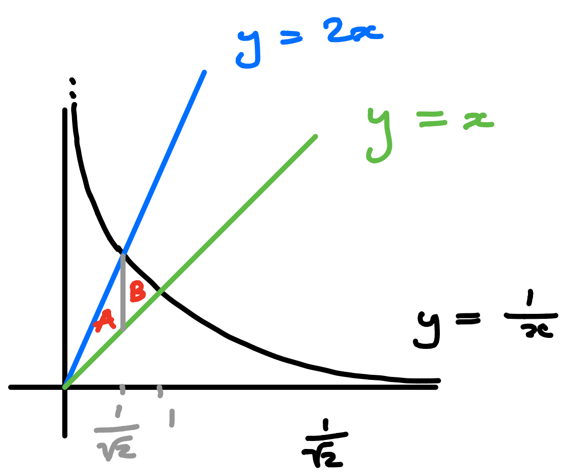
<u>Additional Example</u> : What is the area of the region enclosed by $y = \frac{1}{x}$, $y = x$ and $y = 2x$ in the first quadrant?

<u>Picture</u>

$$2x = \frac{1}{x} \Rightarrow x = \frac{1}{\sqrt{2}}$$

$x > 0$

$$x = \frac{1}{x} \Rightarrow x = 1$$



$$\text{Area}(A) = \int_0^{\frac{1}{\sqrt{2}}} 2x - x \, dx = \frac{1}{2}x^2 \Big|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{4}$$

$$\text{Area}(B) = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{x} - x = \ln|x| - \frac{1}{2}x^2 \Big|_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \left(\ln|1| - \frac{1}{2}\right) - \left(\ln\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{4}\right)$$

$$= \frac{1}{2}\ln(2) - \frac{1}{4}$$

$$\Rightarrow \text{Area enclosed} = \left(\frac{1}{2}\ln(2) - \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{2}\ln(2)$$