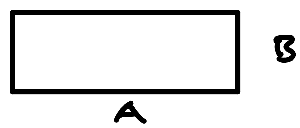
Areas

Q: How can we find the area of a complicated shape?

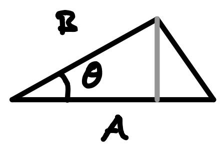
Basic Examples:

Rectangle:



$\Rightarrow$ Area $= AB$

Triangle:



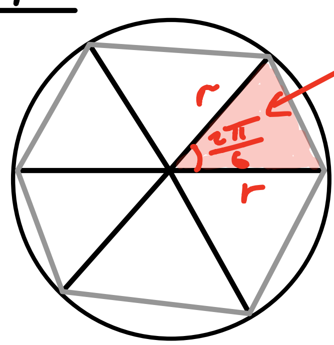
$\Rightarrow$ Height $= B\sin\theta$

Area $= \frac{1}{2} AB\sin\theta$

How about circle of radius r?

Basic Strategy: Approximate circle using triangles to higher and higher accuracy.

Example:



Area $= \frac{1}{2} r^2 \sin\left(\frac{2\pi}{6}\right)$

$\Rightarrow$ Area of circle $\approx \frac{1}{2} r^2 \cdot 6 \sin\left(\frac{2\pi}{6}\right)$

Approximation using n triangles

$$\Rightarrow \text{Area of circle} = \lim_{n\to\infty} \frac{1}{2} r^2 \cdot n \cdot \sin\left(\frac{2\pi}{n}\right)$$

$$= \frac{1}{2} r^2 \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{1/n}$$

L'Hospital $\longrightarrow$

$$= \frac{1}{2} r^2 \lim_{n\to\infty} \frac{\frac{-2\pi}{n^2}\cos\left(\frac{2\pi}{n}\right)}{-1/n^2}$$

$$= \frac{1}{2} r^2 \lim_{n\to\infty} 2\pi \cos\left(\frac{2\pi}{n}\right)$$

Awesome! $\longrightarrow$

$$= \pi r^2$$

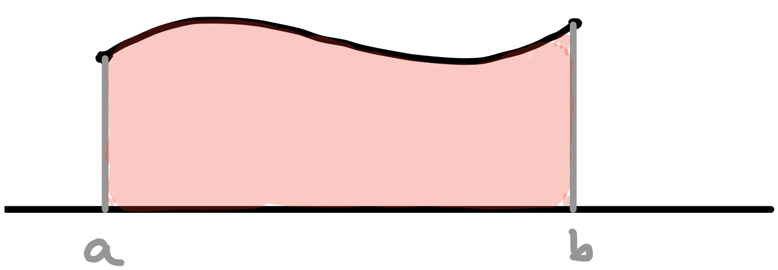
Conclusion: To determine areas of complicated shapes we should approximate using shapes with known area (e.g. triangles and rectangles) and take a limit.

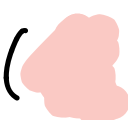
f - non-negative function on $[a,b]$.

Q: Can we calculate the area bounded by $y = f(x)$ and the x-axis over $[a,b]$?

Picture:

$y = f(x)$



Area() = ?

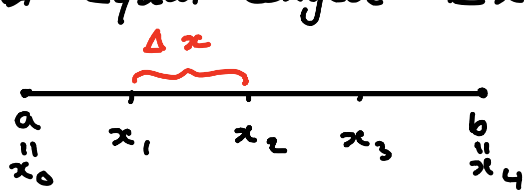
Strategy: Approximate to higher and higher accuracy using rectangles.

Step 1

This will be number of rectangles

Fix n, a natural number, and divide $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$

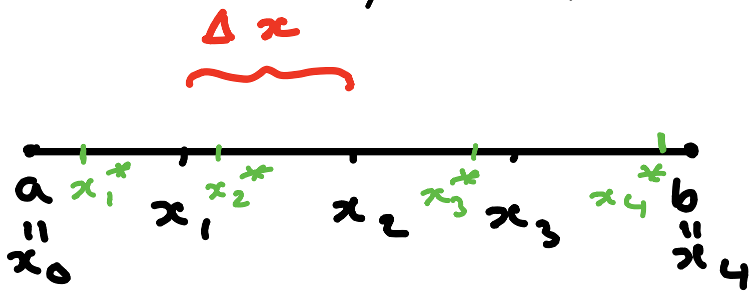
Example $n = 4$



More generally: $x_0 = a$, $x_i = a + i\Delta x$, $x_n = b$

Step 2 Choose a random / sample point in each subinterval.

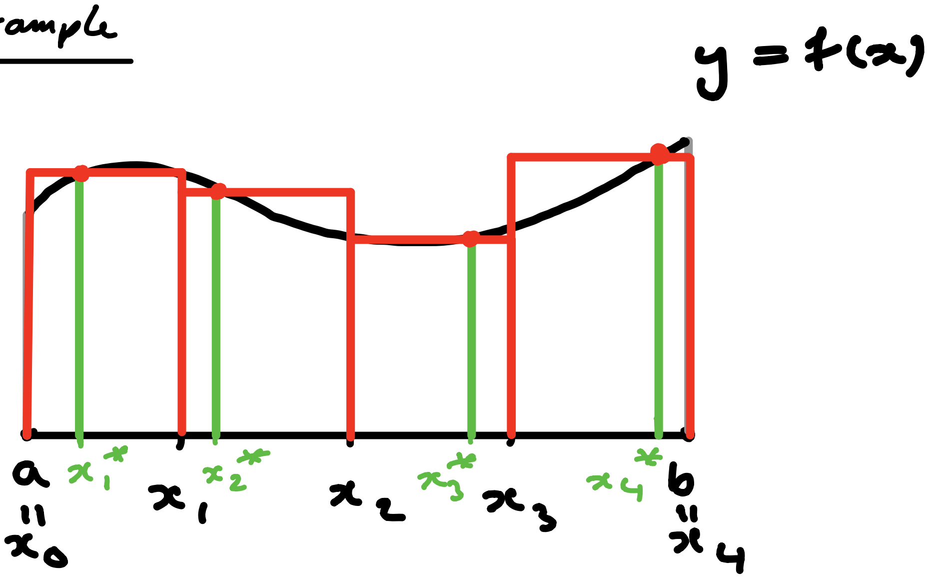
Example



More generally : x_i^* is a sample point in $[x_{i-1}, x_i]$

Step 3 Over each subinterval, draw rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$.

Example



Step 4

Sum the areas of rectangles and take limit as $n \to \infty$

Example $n = 4 \Rightarrow$

Area of rectangle over $[x_2, x_3]$

Sum of Areas of Rectangles

$$= f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

$$= \sum_{i=1}^{4} f(x_i^*)\Delta x$$

Sigma notation for sum

$$a_1 + a_2 + a_3 + \dots + a_n = \sum_{i=1}^{n} a_i$$

where count stops

i^{th} term of sum

where count starts

More generally:

Area under $y = f(x)$ between $x = a$ and $x = b$ $= \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x}_{\text{Called a Riemann Sum, denoted } R_n}$

Basically a precise definition of area under a curve

Remark

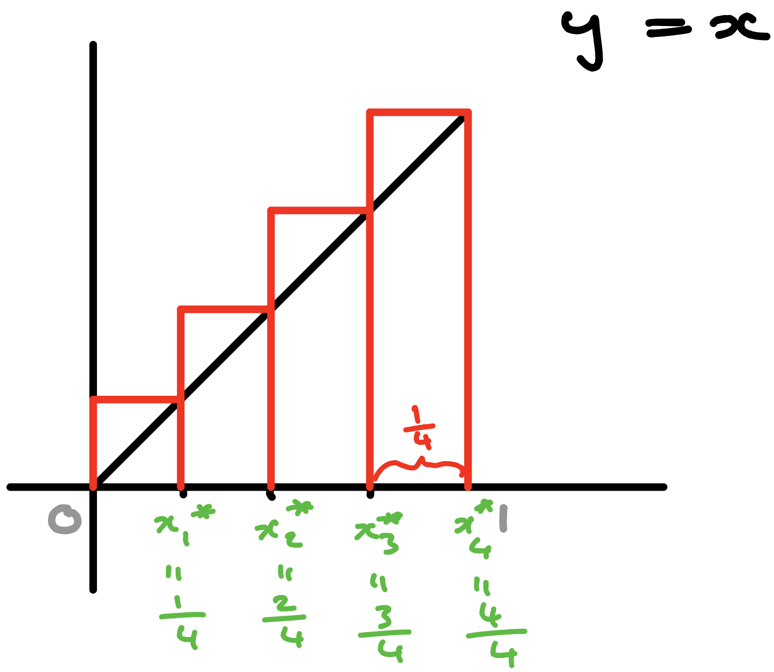
1/ The area should <u>not</u> depend on our choices of x_i^*. For example, we could choose $x_i^* = x_{i-1}$ or $x_i^* = x_i$ and we'd still get same answer.

2/ This is a highly conceptual definition for most functions it will be very hard to carry this computation out.

Example $f(x) = x$, $a = 0$, $b = 1$, $x_i^* = x_i$

$\uparrow$ Our choice

Picture $(n = 4) \Rightarrow \Delta x = \frac{1}{4}$



$$\Rightarrow R_4 = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

$$= \frac{1}{4}\,\frac{1}{4} + \frac{2}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \frac{4}{4}\cdot\frac{1}{4}$$

More generally:

$$R_n = \frac{1}{n}\cdot\frac{1}{n} + \frac{2}{n}\cdot\frac{1}{n} + \dots + \frac{n}{n}\cdot\frac{1}{n}$$

$$= (1 + 2 + 3 + \dots + n)\cdot\frac{1}{n^2}$$

Clever Observation: $\leftarrow$ Discovered by Gauss as an elementary school student!

If $s_n = 1 + 2 + \cdots + n$

$$\Rightarrow 2s_n = s_n + s_n = \begin{array}{l} 1 + 2 + 3 + \cdots + n \\ + n + n-1 + \cdots + 1 \end{array}$$

$$= (n+1) \times n$$

$$\Rightarrow s_n = \frac{n(n+1)}{2}$$

$$\Rightarrow R_n = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{n^2+n}{2n^2}$$

$\Rightarrow$ Area under $y = x$ between $x = 0$ and $x = 1$ $= \lim_{n \to \infty} \dfrac{n^2+n}{2n^2} = \dfrac{1}{2}$

same degree top and bottom

Conclusion :

It's a beautiful conceptual definition, but we'll need a less direct method to calculate it in general.