Antiderivatives

F, f - functions

$F'(x) = f(x) \Rightarrow f$ is the derivative of F.

Alternate Perspective : F is an <u>antiderivative</u> of f.

<u>Examples</u>

1) Velocity = Derivative of position (with respect to time).

$\Rightarrow$ Position = An antiderivative of velocity.

2) $\frac{d}{dx}(x^2) = 2x \Rightarrow x^2$ an antiderivative of $2x$.

3) $\frac{d}{dx} \ln|x| = \frac{1}{x} \Rightarrow \ln|x|$ an antiderivative of $\frac{1}{x}$.

<u>Important Observation</u> : Antiderivatives are <u>not unique</u>.

$$\frac{d}{dx}(\sin(x)) = \frac{d}{dx}(\sin(x) + 1) = \cos(x)$$

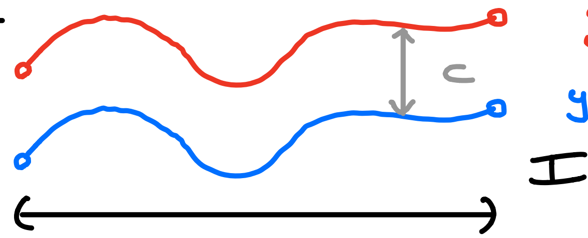
$\Rightarrow$ Both $\sin(x)$ <u>and</u> $\sin(x)+1$ are antiderivatives of $\cos(x)$.

<u>Recall</u> :

$F(x)$, $G(x)$ differentiable on open interval I and $F'(x) = G'(x)$ for all x in I $\Rightarrow$ $G(x) = F(x) + C$ for all x in I

constant

Picture



Conclusion: On open intervals antiderivatives are unique up to addition of a constant.

Example

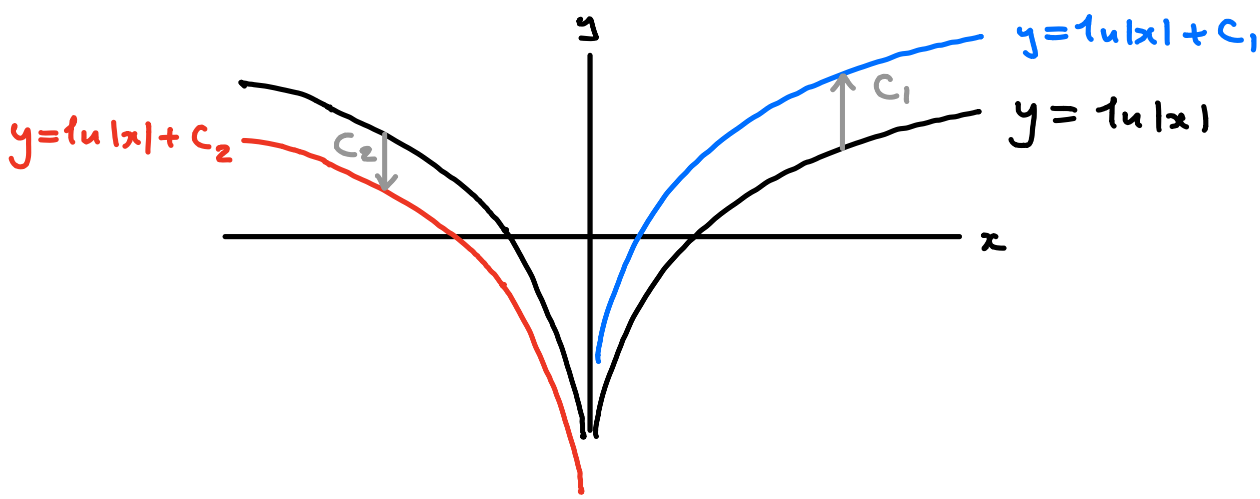
1/ $\frac{d}{dx}(x^3) = 3x^2$ on $\overset{I}{\overset{\shortparallel}{(-\infty,\infty)}} \Rightarrow$ $x^3 + C$ is most general antiderivative of $3x^2$ on $(-\infty,\infty)$

Two intervals

2/ $\frac{d}{dx} \ln|x| = \frac{1}{x}$ on $(-\infty,0) \cup (0,\infty)$

$$\Rightarrow \quad F(x) = \begin{cases} \ln|x| + C_1 & \text{if } x > 0 \\ \ln|x| + C_2 & \text{if } x < 0 \end{cases}$$

most general antiderivative of $1/x$ on $(-\infty,0)\cup(0,\infty)$.



Warning: Finding antiderivatives is much more difficult than finding derivatives. This is because of the unexpected complexity of the product and chain rules.

Example Can we find an antiderivative of e^{x^2}?

Naive Guess:

$$\frac{d}{dx}\left(e^{x^2}\right) = 2x\, e^{x^2} \Rightarrow \frac{1}{2x} e^{x^2} \text{ an antiderivative of } e^{x^2}$$

Very wrong. We've forgotten product rule.

Product Rule

$$\frac{d}{dx}\left(\frac{1}{2x} e^{x^2}\right) = \frac{-1}{2x^2} e^{x^2} + e^{x^2}$$

Not zero.

Crazy Fact: Even though e^{x^2} does have an antiderivative on $(-\infty, \infty)$, it cannot be expressed in terms of our core functions.

Remark The above approach failed because $\frac{d}{dx}(x^2) = 2x$ is <u>non-constant</u>. If it had been constant it would have worked. For example:

$$\frac{d}{dx}(3x+2) = 3$$

Constant multiple rule

$$\frac{1}{3} \text{ constant} \Rightarrow \frac{d}{dx}\frac{1}{3} e^{(3x+2)} = \frac{1}{3}\frac{d}{dx} e^{(3x+2)} = e^{(3x+2)}$$

$$\Rightarrow \frac{1}{3} e^{(3x+2)} \text{ an antiderivative of } e^{(3x+2)}$$

More Generally :

$$F(x) \text{ an antiderivative of } f(x) \Rightarrow \frac{1}{a}F(ax+b) \text{ an antiderivative of } f(ax+b)$$

Constants

This only works because $\frac{d}{dx}(ax+b) = a$ is constant.

G - antiderivative of g

F - antiderivative of f

$G'(x) = g(x)$

$F'(x) = f(x)$

Basic Antiderivative Rules :

Function	Antiderivative
$f + g$	$F + G$
cf	cF
$f(ax+b)$	$\frac{1}{a}F(ax+b)$

Sum rule in reverse

Constant Multiple rule in reverse

constant

Must be linear

The product / Chain rule are much more complicated so cannot be reversed easily. We'll do the Chain Rule later and the product rule in 1B.

Core Antiderivatives

Function	Antiderivative	
x^r	$\frac{1}{r+1}x^{(r+1)}$	$r \neq -1$
$1/x$	$\ln\lvert x\rvert$	
b^x	$\frac{1}{\ln(b)}b^x$	$b>0$, $b \neq 1$
$\sin(x)$	$-\cos(x)$	
$\cos(x)$	$\sin(x)$	
$\sec^2(x)$	$\tan(x)$	
$\frac{1}{\sqrt{1-x}}$ / $\frac{-1}{\sqrt{1-x}}$	$\arcsin(x)$ / $\arccos(x)$	
$\frac{1}{1+x^2}$	$\arctan(x)$	

Observation :

F, G antiderivatives of f on interval I

and $G(a) = F(a)$ for some a in I

constant

$\Rightarrow \quad G(x) = F(x) + C$ for all x in I

$\Rightarrow \quad G(a) = F(a) + C \Rightarrow C = 0$

$\Rightarrow \quad G = F$ on I

Conclusion: If we specify $F(a)$ for some a in I then F is <u>unique</u> antiderivative with this property.

Example

acceleration

$a(t) = 7t^2$.

velocity position

If $v(1) = 2$ and $s(0) = 2$ determine $s(t)$

$v'(t) = a(t) \Rightarrow v'(t) = 7t^2 \Rightarrow v(t) = \frac{7}{3}t^3 + C_1$

$v(1) = 2 \Rightarrow \frac{7}{3} + C_1 = 2 \Rightarrow C_1 = -\frac{1}{3} \Rightarrow v(t) = \frac{7}{3}t^3 - \frac{1}{3}$

$s'(t) = v(t) \Rightarrow s(t) = \frac{7}{12}t^4 - \frac{1}{3}t + C_2$

$s(0) = 2 \Rightarrow C_2 = 2 \Rightarrow s(t) = \frac{7}{12}t^4 - \frac{1}{3}t + 2$