HW 9 Solutions

§11.1

1/
$$\begin{aligned} f(x) &= \sin(x) \\ f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \end{aligned} \Rightarrow \begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \end{aligned} \Rightarrow P_3(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3$$
$$= x - \frac{x^3}{3!}$$

4/
$$\begin{aligned} f(x) &= 5e^{2x} \\ f'(x) &= 10e^{2x} \\ f''(x) &= 20e^{2x} \\ f'''(x) &= 40e^{2x} \end{aligned} \Rightarrow \begin{aligned} f(0) &= 5 \\ f'(0) &= 10 \\ f''(0) &= 20 \\ f'''(0) &= 40 \end{aligned} \Rightarrow P_3(x) = 5 + \frac{10}{1!}x + \frac{20}{2!}x^2 + \frac{40}{3!}x^3$$

6/
$$\begin{aligned} f(x) &= (x+2)^{-1} \\ f'(x) &= -(x+2)^{-2} \\ f''(x) &= 2(x+2)^{-3} \\ f'''(x) &= -6(x+2)^{-4} \end{aligned} \Rightarrow \begin{aligned} f(0) &= 2^{-1} \\ f'(0) &= -2^{-2} \\ f''(0) &= 2\cdot 2^{-3} \\ f'''(0) &= -6\cdot 2^{-4} \end{aligned} \Rightarrow P_3(x) = 2^{-1} + \frac{-2^{-2}}{1!}x + \frac{2\cdot 2^{-3}}{2!}x^2 + \frac{-6\cdot 2^{-4}}{3!}x^3$$

8/
$$\begin{aligned} f(x) &= (1-x)^{\frac{1}{2}} \\ f'(x) &= -\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \\ f''(x) &= \tfrac{-1}{4}(1-x)^{-\frac{3}{2}} \\ f'''(x) &= \tfrac{-3}{8}(1-x)^{-\frac{5}{2}} \end{aligned} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= -\tfrac{1}{2} \\ f''(0) &= \tfrac{-1}{4} \\ f'''(0) &= \tfrac{-3}{8} \end{aligned} \Rightarrow P_3(x) = 1 + \frac{-\frac{1}{2}}{1!}x + \frac{\frac{-1}{4}}{2!}x^2 + \frac{\frac{-3}{8}}{3!}x^3$$

10/
$$f(x) = \ln(1-x) \qquad f(0) = 0$$
$$f'(x) = \frac{-1}{1-x} \qquad f'(0) = -1$$
$$f''(x) = \frac{-1}{(1-x)^2} \qquad f''(0) = -1$$
$$f'''(x) = \frac{-2}{(1-x)^3} \quad \Rightarrow \quad f'''(0) = -2 \quad \Rightarrow \quad P_4(x) = -x - \frac{x^2}{2} - \frac{2}{3!}x^3 - \frac{6}{4!}x^4$$
$$f^{(4)}(x) = \frac{-6}{(1-x)^4} \qquad f^{(4)}(0) = -6 \qquad = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$$\ln(1-x) \approx P_4(x) \Rightarrow \overset{\ln(0.9)}{\overset{\parallel}{\ln(1-0.1)}} \approx P_4(0.1) = -0.1 - \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}.$$

14/
$$f(x) = x^2+2x+1 \qquad f(0) = 1$$
$$f'(x) = 2x+2 \qquad f'(0) = 2$$
$$f''(x) = 2 \quad \Rightarrow \quad f''(0) = 2 \quad \Rightarrow \quad P_n(x) = \begin{cases} 1+2x & \text{if } n = 1 \\ 1+2x+x^2 & \text{if } n=2 \\ 1+2x+x^2 & \text{if } n>2 \end{cases}$$
$$f'''(x) = 6 \qquad \vdots$$
$$\vdots \qquad f^{(n)}(0) = 0$$
$$f^{(n)}(x) = 0$$

15/
$$f(x) = \ln(1+x^2)$$
$$f'(x) = \frac{2x}{1+x^2} \qquad f(0) = 6$$
$$\Rightarrow f'(0) = 0 \quad \Rightarrow \quad P_2(x) = \frac{2}{2!}x^2 = x^2$$
$$f''(x) = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} \qquad f''(0) = 2$$

$$\Rightarrow \int_0^{1/2} \ln(1+x^2)\,dx \approx \int_0^{1/2} x^2\,dx = \left.\frac{x^3}{3}\right|_0^{1/2} = \frac{1}{24}$$

Q17/

$$f(x) = \frac{1}{5-x} \qquad f(4) = 1$$

$$f'(x) = \frac{1}{(5-x)^2} \qquad f'(4) = 1$$

$$f''(x) = \frac{2}{(5-x)^3} \quad \Rightarrow \quad f''(4) = 2 \quad \Rightarrow \quad P_3(x) = 1 + (x-4) + (x-4)^2 + (x-4)^3$$

$$f'''(x) = \frac{6}{(5-x)^4} \qquad f'''(4) = 6$$

Q18/

$$f(x) = \ln(x) \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1$$

$$f'''(x) = \frac{+2}{x^3} \quad \Rightarrow \quad f'''(1) = 2 \quad \Rightarrow \quad P_4(x) = \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$f^{(4)}(x) = \frac{-6}{x^4} \qquad f^{(4)}(1) = -6$$

$$+ \frac{-6}{4!}(x-1)^4$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

21/

$$f(x) = x^{\frac{1}{2}} \qquad f(9) = 3$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow f'(9) = \frac{1}{6} \Rightarrow P_2(x) = 3 + \frac{1}{6}(x-9) + \frac{\frac{-1}{108}}{2!}(x-9)^2$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \qquad f''(9) = \frac{-1}{108}$$

$$\underset{\substack{\uparrow \\ x \text{ near } 9}}{\sqrt{x} \approx} P_2(x) \Rightarrow \sqrt{9.3} \approx P_2(9.3) = 3 + \frac{1}{6}\cdot\frac{3}{10} + \frac{-1}{216}\left(\frac{3}{10}\right)^2$$

23/

$$f(x) = x^4 + x + 1 \qquad f(2) = 19$$

$$f'(x) = 4x^3 + 1 \qquad f'(2) = 33$$

$$f''(x) = 12x^2 \qquad f''(2) = 48$$

$$f'''(x) = 24x \Rightarrow f'''(2) = 48$$

$$f^{(4)}(x) = 24 \qquad f^{(4)}(2) = 24$$

$$\vdots$$

$$f^{(n)}(x) = 0 \qquad f^{(n)}(2) = 0$$

$$\Rightarrow P_n(x) = \begin{cases} 19 + \frac{33}{1!}(x-2) & \text{if } n=1 \\ 19 + \frac{33}{1!}(x-2) + \frac{48}{2!}(x-2)^2 & \text{if } n=2 \\ 19 + \frac{33}{1!}(x-2) + \frac{48}{2!}(x-2)^2 + \frac{48}{3!}(x-2)^3 & \text{if } n=3 \\ 19 + \frac{33}{1!}(x-2) + \frac{48}{2!}(x-2)^2 + \frac{48}{3!}(x-2)^3 + \frac{24}{4!}(x-2)^4 & \text{if } n \geq 4 \end{cases}$$

24/

$$f(x) = x^{-1} \qquad f(1) = 1$$

$$f'(x) = (-1)\cdot(x^{-2}) \qquad f'(1) = -1$$

$$f''(x) = (-1)(-2)\,x^{-3} \Rightarrow f''(1) = (-1)^2\cdot 2!$$

$$f'''(x) = (-1)(-2)(-3)\,x^{-4} \qquad f'''(1) = (-1)^3\cdot 3!$$

$$\vdots$$

$$f^{(n)}(x) = (-1)(-2)\cdots(-n)\,x^{-(n+1)} \qquad f^{(n)}(1) = (-1)^n n!$$

$$\Rightarrow P_n(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots + (-1)^n\cdot(x-1)^n$$

§11.3

1/ $a = 1$, $r = \frac{1}{6}$ $\Rightarrow$ convergent with sum $\dfrac{1}{1-\frac{1}{6}}$

3/ $a = 1$, $r = \frac{-1}{3^2}$ $\Rightarrow$ convergent with sum $\dfrac{1}{1-\left(\frac{-1}{3^2}\right)}$

5/ $a = 2$, $r = \frac{1}{3}$ $\Rightarrow$ convergent with sum $\dfrac{2}{1-\frac{1}{3}}$

13/ $a = 5$, $r = \frac{4}{5}$ $\Rightarrow$ convergent with sum $\dfrac{5}{1-\frac{4}{5}}$

21/ $0.999\ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$ $\leftarrow$ geometric with $a = \frac{9}{10}$, $r = \frac{1}{10}$

$\Rightarrow 0.999\ldots = \dfrac{\frac{9}{10}}{1-\frac{1}{10}} = 1$

23/ $MPC = 0.95$

Extra spending $= 10{,}000{,}000{,}000 \times 0.95 + 10{,}000{,}000{,}000 \times (0.95)^2 + \ldots$

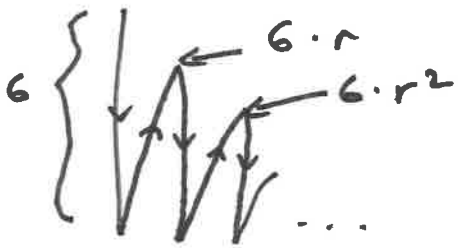
(geometric with $a = 10{,}000{,}000{,}000 \times 0.95$
$r = 0.95$)

$= 10{,}000{,}000{,}000 \cdot \dfrac{0.95}{1-0.95}$

24/

$$\text{Extra spending} = \overset{\substack{\text{billion} \\ \downarrow}}{20} \times 0.98 + 20 \times (0.98)^2 + \dots = 20 \cdot \underbrace{\frac{0.98}{1 - 0.98}}_{\text{Multiplier}}$$

28/ r = Coefficient of restitution



$\Rightarrow$ Total Distance Travelled

$$= 6 + 2 \cdot 6 \cdot \overset{\substack{\text{up and down} \\ \downarrow}}{r} + 2 \cdot 6 \cdot r^2 + 2 \cdot 6 \cdot r^3 + \dots$$

$$= 6 + 12r + 12r^2 + 12r^3 + \dots$$

$$= -6 + \underbrace{12 + 12r + 12r^2 + \dots}_{\text{geometric}}$$

$$a = 12$$

$$= -6 + \frac{12}{1-r}$$

Tennis ball $\Rightarrow r = \frac{7}{10} \Rightarrow$ Total distance travelled $= -6 + \dfrac{12}{1 - \frac{7}{10}}$

29/ After 1st dose $\longrightarrow 6$

After 2nd dose $\longrightarrow 6 + 0.7 \cdot 6$

After 3rd dose $\longrightarrow 6 + 6 \cdot (0.7) + 6 \cdot (0.7)^2$

$\Big\}$ $a = 6$, $r = 0.7$

$\dots \Rightarrow$ In long term dose will be $\dfrac{6}{1 - 0.7}$ mg.

30/

$$\left.\begin{array}{l} \text{Before } 1^{st} \text{ dose} \longrightarrow 0 \\ \text{Before } 2^{nd} \text{ dose} \longrightarrow (0.8).2 \\ \text{Before } 3^{rd} \text{ dose} \longrightarrow (0.8).2 + (0.8)^2.2 \\ \text{Before } 4^{th} \text{ dose} \longrightarrow (0.8).2 + (0.8)^2.2 + (0.8)^3.2 \end{array}\right\} \begin{array}{l} a = (0.8).2 \\ r = 0.8 \end{array}$$

$\Rightarrow$ In long term dose will be $\dfrac{(0.8).2}{1-0.8}$ mg.

31/ In long term, dose immediately after a dose $= \dfrac{M}{1-0.75} \approx 20$ mg

$\Rightarrow$ $M = 20(1-0.75) = 5$ mg.

40/ $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{2k} = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^6 + \dots$ ($\Rightarrow$ geometric with $a = \left(\frac{1}{3}\right)^2$, $r = \left(\frac{1}{3}\right)^2$)

$$= \frac{\left(\frac{1}{3}\right)^2}{1-\left(\frac{1}{3}\right)^2}.$$

42/

more blocks

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{1}{4}+\frac{1}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9}+\dots+\frac{1}{16}\right)}_{>8\times\frac{1}{16} = \frac{1}{2}} + \left(\dots \dots \frac{1}{2^k}\right)$$

$$\Rightarrow \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \geqslant 1 + n \times \frac{1}{2} = 1 + \frac{n}{2}$$

n brackets each $\geqslant \frac{1}{2}$

The right hand side grows positively without as n grows, hence the left hand side does too.

$\Rightarrow \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is <u>divergent</u>.