Homework 7

§10.1

Q1/ $f'(t) = 3t\,e^{t^2} \Rightarrow f'(t) - 2t\,f(t) = 3te^{t^2} - 2t\left(\frac{3}{2}e^{t^2} - \frac{1}{2}\right)$

$= t$

Q2/ $f'(t) = 2t \Rightarrow (f'(t))^2 - 4f(t) = (2t)^2 - 4\left(t^2 - \frac{1}{2}\right)$

$= 2.$

Q9/ $y' = t^2(y-5) \Rightarrow y(t) = 5$ is a constant solution.

Q13 $v'(t) = 0.2 \cdot (160 - v(t))$

$v(t) = 60 \Rightarrow v'(t) = 0.2 \cdot (160 - 60) = 20.\ ft/s^2.$

Q16/ a) $y(t)$ = account balance at time t

$y(1) = 10000$

$\Rightarrow y'(1) = (0.04) \cdot 10000 + 2000 > 0 \Rightarrow$ increasing.

b) $y' = 0.04y + 2000 = 0.04(y + 50000)$

c) The accout balance changes at a rate which is directly proportional to the balance plus 50000.

Q19/ $y' = k(C - y)$

$\uparrow$ constant.

§10.2

Q1/ $\dfrac{dy}{dt} = \dfrac{5-t}{y^2} \Rightarrow$ No constant sols as $\dfrac{1}{y^2}$ has no zeros.

$\Rightarrow \int y^2\,dy = \int (5-t)\,dt \Rightarrow \dfrac{1}{3}y^3 = -\dfrac{(5-t)^2}{2} + C$

$\Rightarrow y = \sqrt[3]{-\dfrac{3(5-t)^2}{2} + 3C}$ is genl solution.

(C arbitrary constant)

Q6/ $\frac{dy}{dt} = \frac{t^2}{t^3+8} \cdot y^2$

$y^2 = 0 \Rightarrow y = 0 \Rightarrow y=0$ only constant solution.

$\int \frac{1}{y^2}\, dy = \int \frac{t^2}{t^3+8}\, dt$

$u = t^3+8 \Rightarrow \frac{du}{dt} = 3t^2 \Rightarrow dt = \frac{du}{3t^2}$

$\Rightarrow \int \frac{t^2}{t^3+8}\, dt = \frac{1}{3}\int \frac{1}{u}\, du = \frac{1}{3}\ln|u| + c$

$= \frac{1}{3}\ln|t^3+8| + c$

$\Rightarrow \frac{-1}{y} = \frac{1}{3}\ln|t^3+8| + c \Rightarrow y = \frac{-1}{\frac{1}{3}\ln|t^3+8|+c}$

$\Rightarrow y = \begin{cases} 0 \\ \frac{-1}{\frac{1}{3}\ln|t^3+8|+c} \end{cases}$ c arbitrary

Q10/ $\dfrac{dy}{dt} = e^{2t} \cdot \dfrac{1}{y^2} \Rightarrow$ No constant solutions as $\dfrac{1}{y^2} \neq 0$.

$\int y^2 dy = \int e^{2t} dt$

$\Rightarrow \dfrac{1}{3} y^3 = \dfrac{1}{2} e^{2t} + C \Rightarrow y = \sqrt[3]{\dfrac{3}{2} e^{2t} + 3C}$

Q15 $\dfrac{dy}{dt} = \dfrac{\ln(t)}{t} \cdot \dfrac{1}{y} \Rightarrow$ No constant solutions as $\dfrac{1}{y} \neq 0$

$\int y \, dy = \int \dfrac{\ln(t)}{t} dt$

$u = \ln(t) \Rightarrow \dfrac{du}{dt} = \dfrac{1}{t} \Rightarrow dt = t\,du \Rightarrow \int \dfrac{\ln(t)}{t} dt = \int u\,du$

$= \dfrac{1}{2} u^2 + C = \dfrac{1}{2} (\ln(t))^2 + C$

$\Rightarrow \dfrac{1}{2} y^2 = \dfrac{1}{2} (\ln(t))^2 + C \Rightarrow y = \pm \sqrt{(\ln(t))^2 + 2C}$

Q18/ $y^2 y' = \tan(t) \Rightarrow \frac{dy}{dt} = \tan(t) \cdot \frac{1}{y^2}$ ← never 0 so no constant solutions

$$\int y^2 \, dy = \int \tan(t) \, dt = \int \frac{\sin(t)}{\cos(t)} \, dt$$

$u = \cos(t) \Rightarrow \frac{du}{dt} = -\sin(t) \Rightarrow dt = \frac{du}{-\sin(t)}$

$$\Rightarrow \int \frac{\sin(t)}{\cos(t)} \, dt = -\int \frac{1}{u} \, du = -\ln|u| + C$$

$$= -\ln|\cos(t)| + C$$

$$\Rightarrow \frac{1}{3} y^3 = -\ln|\cos(t)| + C \Rightarrow y = \sqrt[3]{-3\ln|\cos(t)| + 3C}$$

Q26 $\frac{dy}{dt} = (1+t)^2 \cdot \frac{1}{(1+y)^2}$ ← never 0 so no constant solutions.

$$\int (1+y)^2 \, dy = \int (1+t)^2 \, dt \Rightarrow \frac{1}{3}(1+y)^3 = \frac{1}{3}(1+t)^3 + C$$

$$\Rightarrow \quad y = -1 + \sqrt[3]{(1+t)^3 + 3C}$$

$$y(0) = 2 \Rightarrow -1 + \sqrt[3]{1+3C} = 2 \Rightarrow 1+3C = 27 \Rightarrow C = \frac{26}{3}$$

$$\Rightarrow \quad y = -1 + \sqrt[3]{(1+t)^3 + 26}$$

Q24

$$\frac{dy}{dx} = \frac{\ln(x)}{\sqrt{x}} \cdot \frac{1}{\sqrt{y}}$$

never 0 so no constant solutions

$$\int \sqrt{y}\, dy = \int \frac{\ln(x)}{\sqrt{x}}\, dx$$

$f(x) = \ln(x)$ $\quad g(x) = \frac{1}{\sqrt{x}}$

$f'(x) = \frac{1}{x}$ $\quad G(x) = 2\sqrt{x}$

$$\Rightarrow \int \frac{\ln(x)}{\sqrt{x}}\, dx = 2\sqrt{x}\ln(x) - 2\int \frac{1}{\sqrt{x}}\, dx$$

$$= 2\sqrt{x}\ln(x) - 4\sqrt{x} + C$$

$$\Rightarrow \quad \frac{2}{3} y^{3/2} = 2\sqrt{x}\ln(x) - 4\sqrt{x} + C$$

$$\Rightarrow \quad y = \left(3\sqrt{x}\ln(x) - 6\sqrt{x} + \frac{3}{2}C\right)^{2/3}$$

$y(1) = 4$

$\Rightarrow \quad \left(-6 + \frac{3}{2}c\right)^{2/3} = 4 \quad \Rightarrow \quad -6 + \frac{3}{2}c = 8 \quad \Rightarrow \quad \frac{3}{2}c = 14$

$\Rightarrow \quad y = \left(3\sqrt{x}\ln(x) - 6\sqrt{x} + 14\right)^{2/3}.$

Q31 $\quad \frac{dy}{dp} = -\frac{1}{2} \cdot \frac{1}{p+3} \cdot y$

$\Rightarrow y = 0$ is constant Solution. $\quad p \geq 0$

$\int \frac{1}{y} dy = \int -\frac{1}{2} \cdot \frac{1}{p+3} dp \quad \Rightarrow \quad \ln|y| = -\frac{1}{2}\ln|p+3| + C$

$\Rightarrow \quad \ln|y| = \ln\left((p+3)^{-\frac{1}{2}}\right) + c$

$\Rightarrow \quad y = \pm e^{c} \cdot (p+3)^{-\frac{1}{2}}$

$\Rightarrow$ General sol is $y = \begin{cases} 0 \\ \pm e^{c} \cdot \frac{1}{\sqrt{p+3}} \end{cases}$

$\Rightarrow \quad y = \frac{A}{\sqrt{p+3}}$ for A any constant

Q32 $\frac{dy}{ds} = k \cdot \frac{1}{s} \cdot y \quad (s \geqslant 0)$

$\Rightarrow y = 0$ constant solution

$\int \frac{1}{y}\, dy = \int \frac{k}{s}\, ds \Rightarrow \ln|y| = k \ln|s| + C = \ln|s|^k + c$

$\Rightarrow y = \pm e^c \cdot |s|^k = \pm e^c \cdot s^k$

$\Rightarrow y = \begin{cases} 0 \\ \pm e^c \cdot s^k \end{cases} \Rightarrow y = A \cdot s^k$ A any constant.

Q35 $\frac{dV}{dt} = k V^{2/3}$ (k some constant)

$\Rightarrow V = 0$ constant sol.

$\int V^{-2/3}\, dV = \int k\, dt \Rightarrow 3 V^{1/3} = kt + c$

$\Rightarrow V = \left(\frac{k}{3} t + \frac{c}{3}\right)^3 = \frac{(kt + c)^3}{27}$

$v(0) = 27 \Rightarrow \dfrac{c^3}{27} = 27 \Rightarrow c = 9$

$v(4) = 15.625 \Rightarrow \dfrac{(4k+9)^3}{27} = 15.625 \Rightarrow k = \dfrac{(27 \cdot (15.625))^{\frac{1}{3}} - 9}{4}$

$$\Rightarrow v(t) = \frac{\left(\left(\frac{(27 \cdot (15.625))^{1/3} - 9}{4}\right) t + 9\right)^3}{27} = \left(3 - \tfrac{1}{8}t\right)^3$$

$$v(t) = 0 \Rightarrow t = \frac{-9}{\left(\frac{27 \cdot (15.625)^{1/3} - 9}{4}\right)} = 24$$

§10.3

Q1/ $y' - 2y = t \Rightarrow a(t) = -2, \; b(t) = t \Rightarrow I(t) = e^{-2t}.$

$A(t) = -2t$

Q5/ $y' - \dfrac{y}{10+t} = 2 \Rightarrow a(t) = \dfrac{-1}{10+t} \quad (t>0)$

$\Rightarrow A(t) = -\ln(10+t) = \ln\left(\dfrac{1}{10+t}\right)$

$= I(t) = e^{\ln\left(\frac{1}{10+t}\right)} = \dfrac{1}{10+t}$

Q6/ $y' = t^2(y+1) \Rightarrow y' + (-t^2)y = t^2 \Rightarrow a(t) = -t^2$

Let $A(t) = -\frac{1}{3}t^3 \Rightarrow I(t) = e^{-\frac{1}{3}t^3}.$

Q13 $y' + \dfrac{y}{10+t} = 0 \Rightarrow a(t) = \dfrac{1}{10+t} \quad (t>0)$

$b(t) = 0$

Let $A(t) = \ln(10+t) \Rightarrow I(t) = e^{\ln(10+t)} = 10+t$

$\Rightarrow y = \dfrac{1}{10+t} \cdot \int 0 \, dt = \dfrac{c}{10+t}.$

Q16/ $y' = e^{-t}(y+1) \Rightarrow y' + (-e^{-t})y = e^{-t}$

$\Rightarrow a(t) = -e^{-t}$, $b(t) = e^{-t}$ $\quad A(t) = e^{-t}$

$\Rightarrow y = \dfrac{1}{e^{e^{-t}}} \cdot \int e^{e^{-t}} \cdot e^{-t}\, dt$

$u = e^{-t} \Rightarrow \dfrac{du}{dt} = -e^{-t} \Rightarrow dt = \dfrac{du}{-e^{-t}}$

$\Rightarrow \int e^{e^{-t}} \cdot e^{-t}\, dt = -\int e^{u}\, du = -e^{u} + C = -e^{e^{-t}} + C$

$\Rightarrow y = -1 + \dfrac{c}{e^{e^{-t}}}$

Q20/ $\dfrac{1}{\sqrt{t+1}} y' + y = 1 \Rightarrow y' + \sqrt{t+1}\, y = \sqrt{t+1}$

$\Rightarrow a(t) = \sqrt{t+1}$, $b(t) = \sqrt{t+1}$, $A(t) = \dfrac{2}{3}(t+1)^{3/2}$

$\Rightarrow y = \dfrac{1}{e^{2/3(t+1)^{3/2}}} \cdot \int e^{2/3(t+1)^{3/2}} \cdot \sqrt{t+1}\, dt$

$$u = \tfrac{2}{3}(t+1)^{3/2} \Rightarrow \frac{du}{dt} = \sqrt{t+1} \Rightarrow dt = \frac{du}{\sqrt{t+1}}$$

$$\Rightarrow \int e^{\frac{2}{3}(t+1)^{3/2}} \cdot \sqrt{t+1}\, dt = \int e^u\, du = e^u + c = e^{\frac{2}{3}(t+1)^{3/2}} + c$$

$$\Rightarrow y = 1 + \frac{c}{e^{\frac{2}{3}(t+1)^{3/2}}}.$$

G22

$$y' + \frac{1}{t} y = \frac{\ln(t)}{t} \Rightarrow a(t) = \frac{1}{t} \quad A(t) = \ln(t) \quad (t>0)$$

$$b(t) = \frac{\ln(t)}{t} \quad e^{A(t)} = t$$

$$\Rightarrow y = \frac{1}{t}\int t \cdot \frac{\ln(t)}{t}\, dt = \frac{1}{t}\int \ln(t)\, dt = \frac{1}{t}(t\ln(t) - t + c)$$

$$= \ln(t) + 1 + \frac{c}{t}$$

$$y(e) = 0 \Rightarrow \ln(e) - 1 + \frac{c}{t} = 0 \Rightarrow \frac{c}{t} = 0 \Rightarrow c = 0$$

$$\Rightarrow y = \ln(t) - 1$$

Q23

$$y' + \frac{y}{1+t} = 20 \Rightarrow a(t) = \frac{1}{1+t} \quad A(t) = \ln(1+t) > 0$$

$$b(t) = 20 \quad e^{A(t)} = 1+t$$

$$\Rightarrow y = \frac{1}{1+t}\int (1+t)\cdot 20\, dt = \frac{1}{1+t}(20t + 10t^2 + C)$$

$$y(0) = 10 \Rightarrow C = 10 \Rightarrow y = \frac{1}{1+t}(20t + 10t^2 + 10)$$

Q25

$$y' + y = e^{2t} \Rightarrow a(t) = 1 \quad A(t) = t$$

$$b(t) = e^{2t} \quad e^{A(t)} = e^t$$

$$\Rightarrow y = \frac{1}{e^t}\int e^{3t}\, dt = \frac{1}{e^t}\left(\frac{1}{3}e^{3t} + C\right) = \frac{1}{3}e^{2t} + \frac{C}{e^t}$$

$$y(0) = -1 \Rightarrow \frac{1}{3} + C = -1 \Rightarrow C = \frac{-4}{3}$$

$$\Rightarrow y = \frac{1}{3}e^{2t} - \frac{4}{3e^t}$$

Q28

$$ty' + y = \sin(t) \Rightarrow y' + \frac{1}{t}y = \frac{\sin(t)}{t} \Rightarrow a(t) = \frac{1}{t}$$

$$b(t) = \frac{\sin(t)}{t}$$

$$A(t) = \ln(t) \quad (t > 0), \quad e^{A(t)} = t$$

$$\Rightarrow \quad y(t) = \frac{1}{t}\int \sin(t)\,dt = \frac{1}{t}\left(-\cos(t) + c\right)$$

$$= \frac{-\cos(t)}{t} + \frac{c}{t}$$

$$y\left(\tfrac{\pi}{2}\right) = 0 \Rightarrow \frac{c}{\frac{\pi}{2}} = 0 \Rightarrow c = 0 \Rightarrow y(t) = \frac{-\cos(t)}{t}$$