HW3 Solutions

Q1/ $F(x, y, \lambda) = x^2 + 3y^2 + 10 + 8\lambda - \lambda x - \lambda y$

$\Rightarrow \frac{\partial F}{\partial x} = 2x - \lambda, \quad \frac{\partial F}{\partial y} = 6y - \lambda, \quad \frac{\partial F}{\partial \lambda} = 8 - x - y$

$\begin{matrix} 2x - \lambda = 0 \\ 6y - \lambda = 0 \end{matrix} \Rightarrow \lambda = 2x = 6y \Rightarrow x = 3y$

$8 - x - y = 0 \Rightarrow 8 - 3y - y = 0 \Rightarrow y = 2 \Rightarrow x = 6 \Rightarrow \lambda = 12$

$\Rightarrow (6, 2, 12)$ only solution $\Rightarrow (6, 2)$ is a min.

Q4/ $F(x, y, \lambda) = \frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2} + 3\lambda x - \lambda y - \lambda$

$\Rightarrow \frac{\partial F}{\partial x} = x - 3y + 3\lambda = 0, \quad \frac{\partial F}{\partial y} = -3x + 2y - \lambda = 0, \quad \frac{\partial F}{\partial \lambda} = 3x - y - 1 = 0$

$\begin{matrix} x - 3y + 3\lambda = 0 \\ -3x + 2y - \lambda = 0 \end{matrix} \Rightarrow \lambda = \frac{3y - x}{3} = -3x + 2y \Rightarrow y - \frac{1}{3}x = -3x + 2y$

$\Rightarrow y = \frac{8}{3}x$

$3x - y - 1 = 0 \Rightarrow 3x - \frac{8}{3}x - 1 = 0 \Rightarrow x = 3 \Rightarrow y = 8 \Rightarrow \lambda = 7$

$\Rightarrow (3, 8, 7)$ only solution $\Rightarrow (3,8)$ is min.

Q6/ $F(x,y,\lambda) = x^2 + xy + y^2 - 2x - 5y + \lambda - \lambda x + \lambda y$

$\Rightarrow \dfrac{\partial F}{\partial x} = 2x + y - 2 - \lambda \; , \; \dfrac{\partial F}{\partial y} = x + 2y - 5 + \lambda \; , \; \dfrac{\partial F}{\partial \lambda} = 1 - x + y$

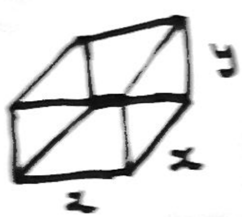
$2x + y - 2 - \lambda = 0$
$x + 2y - 5 + \lambda = 0$ $\Rightarrow$ $\lambda = 2x + y - 2$
$\lambda = 5 - x - 2y$ $\Rightarrow$ $2x + y - 2 = 5 - x - 2y$

$\Rightarrow 3y = -3x + 7 \Rightarrow y = -x + \frac{7}{3}$

$1 - x + y = 0 \Rightarrow 1 - x + (-x + \frac{7}{3}) = 0 \Rightarrow x = \frac{5}{3} \Rightarrow y = \frac{2}{3} \Rightarrow \lambda = 2$

$\Rightarrow (\frac{5}{3}, \frac{2}{3}, 2)$ only solution $\Rightarrow (\frac{5}{3}, \frac{2}{3})$ is minimum location.

Q9/



$f(x,y) = x^2 y$
$g(x,y) = x^2 + 4xy - 300$
$(x, y > 0)$

$\Rightarrow F(x,y,\lambda) = x^2 y + \lambda x^2 + 4xy\lambda - 300\lambda$

$\Rightarrow \dfrac{\partial F}{\partial x} = 2xy + 2\lambda x + 4y\lambda \; , \; \dfrac{\partial F}{\partial y} = x^2 + 4x\lambda \; , \; \dfrac{\partial F}{\partial \lambda} = x^2 + 4xy - 300$

$2xy + 2\lambda x + 4y\lambda = 0$
$x^2 + 4x\lambda = 0$ $\Rightarrow \lambda = \dfrac{-2xy}{2x + 4y} = \dfrac{-x}{4}$

$\Rightarrow \quad -8xy = -2x^2 - 4xy \Rightarrow 2x^2 = 4xy \Rightarrow x = 2y$

$x^2 + 4xy - 300 = 0 \Rightarrow \quad 4y^2 + 8y^2 - 300 = 0 \Rightarrow y^2 = 25 \Rightarrow y = 5$

$\Rightarrow \quad x = 10 \Rightarrow \lambda = \frac{-10}{4}$

$\Rightarrow \quad (10, 5, \frac{-10}{4})$ only solution $\Rightarrow$ $(10,5)$ maximizes volume.

Q10/

$F(x,y,\lambda) = 1000\sqrt{6x^2+y^2} + \lambda \cdot 480 \cdot x + \lambda \cdot 40 \cdot y - 5000 \cdot \lambda$

$\Rightarrow \quad \frac{\partial F}{\partial x} = \frac{1000 \cdot 12 \cdot x}{2\sqrt{6x^2+y^2}} + \lambda \cdot 480 = 0 \qquad \lambda = \frac{-1000 \cdot 12 \cdot x}{480 \cdot 2 \cdot \sqrt{6x^2+y^2}}$

$\Rightarrow$

$\frac{\partial F}{\partial y} = \frac{1000 \cdot 2 \cdot y}{2\sqrt{6x^2+y^2}} + \lambda \cdot 40 = 0 \qquad \lambda = \frac{-1000 \cdot 2 \cdot y}{40 \cdot 2 \cdot \sqrt{6x^2+y^2}}$

$\Rightarrow \quad \frac{12x}{480} = \frac{2y}{40} \Rightarrow x = 2y$

$480x + 40y - 5000 = 0 \Rightarrow 960y + 40y - 5000 = 0 \Rightarrow y = 5 \Rightarrow x = 10$

$\Rightarrow \lambda = \dfrac{-1000 \cdot 2 \cdot 5}{40 \cdot 2 \cdot \sqrt{6 \cdot 10^2 + 5^2}} \Rightarrow$ Minimum must be when $y = 5$ and $x = 10$. This of course assumes there is a minimum.

Q11 $F(x, y, \lambda) = 3x + 4y + 9\lambda x^2 + 4\lambda y^2 - 18000\lambda \Rightarrow$

$\dfrac{\partial F}{\partial x} = 3 + 18\lambda x$, $\dfrac{\partial F}{\partial y} = 4 + 8\lambda y$, $\dfrac{\partial F}{\partial \lambda} = 9x^2 + 4y^2 - 18000$

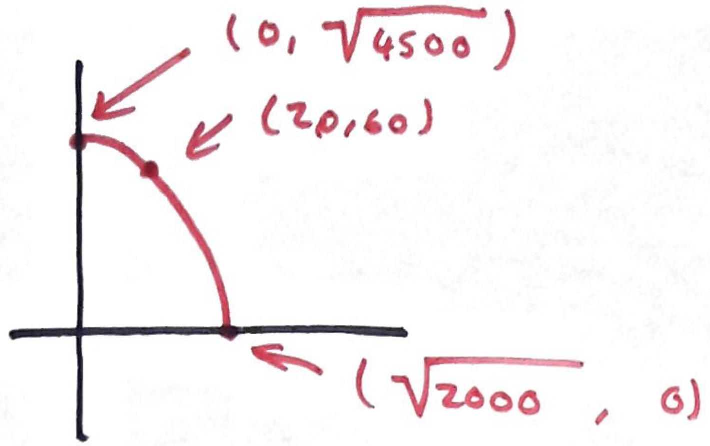
$3 + 18\lambda x = 0$
$4 + 8\lambda y = 0$ $\Rightarrow$ $\lambda = \dfrac{-3}{18x}$, $\lambda = \dfrac{-4}{8y}$ $\Rightarrow \dfrac{-3}{18x} = \dfrac{-4}{8y} \Rightarrow 24y = 72x \Rightarrow y = 3x$

$9x^2 + 4y^2 - 18000 = 0 \Rightarrow 9x^2 + 36x^2 - 18000 = 0 \Rightarrow x^2 = 400 \Rightarrow x = 20$ $(x \geq 0)$

$\Rightarrow y = 60 \Rightarrow \lambda = \dfrac{-3}{18 \cdot 20}$

$\Rightarrow$ Only solution for $x \geq 0$ and $y \geq 0$ is $(20, 60)$

To be sure we have a max must also check endpoints



$P(0,\sqrt{4500}) = 4\cdot\sqrt{4500} \approx 268$

$P(20,60) = 3\cdot 20 + 4\cdot 60 = 300$

$P(\sqrt{2000}, 0) = 3\cdot\sqrt{2000} \approx 134$

$\Rightarrow$ Max profit is when $x=20$, $y=60$.

Q14 $F(x,y,\lambda) = 2x + 10y + 4\lambda x^2 + 25\lambda y^2 - 50000\lambda$

$\Rightarrow \dfrac{\partial F}{\partial x} = 2 + 8\lambda x$, $\dfrac{\partial F}{\partial y} = 10 + 50\lambda y$, $\dfrac{\partial F}{\partial \lambda} = 4x^2 + 25y^2 - 50000$

$2 + 8\lambda x = 0$

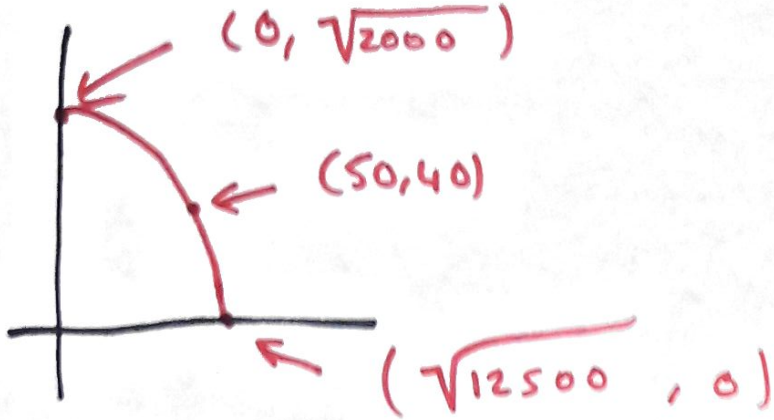
$10 + 50\lambda y = 0$ $\Rightarrow \lambda = \dfrac{-2}{8x} = \dfrac{-10}{50y} \Rightarrow y = \dfrac{4}{5}x$

$4x^2 + 25y^2 - 50000 = 0 \Rightarrow 4x^2 \cdot 25 \cdot \frac{16}{25}x^2 - 50000 = 0 \Rightarrow x^2 = 2500$

$\Rightarrow x = 50 \Rightarrow y = 40 \Rightarrow \lambda = \dfrac{-2}{8\cdot 50}$

$(x \geq 0)$

$\Rightarrow$ $(50, 40)$ is only possible solution with $x > 0,\ y > 0$. Must also check endpoints.



$P(0, \sqrt{2000}) = 10\sqrt{2000} \simeq 447$

$P(50, 40) = 2 \cdot 50 + 10 \cdot 40 = 500$

$P(\sqrt{12500}, 0) = 2 \cdot \sqrt{12500} \approx 223$

$\Rightarrow$ Profit maximized when $x = 50,\ y = 40$.

Q15 a) $F(x, y, \lambda) = 96x + 162y + 64\lambda x^{3/4} y^{1/4} - 3456\lambda$

$\Rightarrow \dfrac{\partial F}{\partial x} = 96 + 48\lambda x^{-1/4} y^{1/4}, \quad \dfrac{\partial F}{\partial y} = 162 + 16\lambda x^{3/4} y^{-3/4}$

$\dfrac{\partial F}{\partial \lambda} = 64x^{3/4} y^{1/4} - 3456$

$96 + 48\lambda x^{-\frac{1}{4}} y^{1/4} = 0 \quad \Rightarrow \quad \lambda = -\dfrac{96}{48} \cdot \dfrac{x^{1/4}}{y^{1/4}} \quad \Rightarrow -2\dfrac{x^{1/4}}{y^{1/4}} = \dfrac{-162}{16} \cdot \dfrac{y^{3/4}}{x^{3/4}}$

$162 + 16\lambda x^{3/4} y^{-3/4} = 0 \qquad \lambda = \dfrac{-162}{16} \cdot \dfrac{y^{3/4}}{x^{3/4}}$

$$\Rightarrow \quad x = \frac{162}{32} y = \frac{81}{16} y = \left(\frac{3}{2}\right)^4 y$$

$$64 x^{3/4} y^{1/4} - 3456 = 0 \Rightarrow 64 \left(\left(\frac{3}{2}\right)^4 y\right)^{3/4} y^{1/4} - 3456 = 0$$

$$\Rightarrow 64 \cdot \frac{27}{8} y - 3456 = 0 \Rightarrow y = 16 \Rightarrow x = 81$$

Assuming a min exists (there are no endpoints to this constraint)

it must be when $y = 16$ and $x = 81$

b) $$\lambda = -2 \cdot \frac{(81)^{1/4}}{(16)^{1/4}} = -2 \cdot \frac{3}{2} = -3 \quad \text{(at optimal)}$$

c) $f(x,y) = 64 x^{3/4} y^{1/4}$, $\frac{\partial f}{\partial x}$ = marginal productivity of labor

$\frac{\partial f}{\partial y}$ = marginal productivity of capital.

$$F(x,y,\lambda) = 96x + 162y + \lambda f(x,y) - 3456\lambda$$

$$\frac{\partial F}{\partial x} = 96 + \lambda \frac{\partial f}{\partial x} \quad , \quad \frac{\partial F}{\partial y} = 162 + \lambda \frac{\partial f}{\partial y}$$

At optimal $\quad 96 + \lambda \frac{\partial f}{\partial x} = 0 = 162 + \lambda \frac{\partial f}{\partial y}$

$$\Rightarrow \lambda = \frac{-96}{\frac{\partial f}{\partial x}} = \frac{-162}{\frac{\partial f}{\partial y}} \Rightarrow \frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{96}{162} .$$

Q19 $\quad F(x,y,z,\lambda) = 3x + 5y + z - x^2 - y^2 - z^2 + 6\lambda - \lambda x - \lambda y - \lambda z$

$$\Rightarrow \frac{\partial F}{\partial x} = 3 - 2x - \lambda \quad , \quad \frac{\partial F}{\partial y} = 5 - 2y - \lambda \quad , \quad \frac{\partial F}{\partial z} = 1 - 2z - \lambda$$

$$\frac{\partial F}{\partial \lambda} = 6 - x - y - z$$

$$\begin{array}{lclcl} 3 - 2x - \lambda = 0 & & \lambda = 3 - 2x & & \\ 5 - 2y - \lambda = 0 & \Rightarrow & \lambda = 5 - 2y & \Rightarrow & \begin{array}{l} 3 - 2x = 5 - 2y \\ 5 - 2y = 1 - 2z \end{array} \\ 1 - 2z - \lambda = 0 & & \lambda = 1 - 2z & & \end{array}$$

$\Rightarrow \quad x = y - 1 \quad$ and $\quad z = y - 2$

$6 - x - y - z = 0 \Rightarrow 6 - (y-1) - y - (y-2) = 0 \Rightarrow 9 - 3y = 0 \Rightarrow y = 3$

$\Rightarrow \quad x = 2 \Rightarrow z = 1$

$\Rightarrow$ (Assuming max exists) max is when $x = 2, y = 3, z = 1$

Q21 $F(x, y, z, \lambda) = 3xy + 2xz + 2yz + \lambda xyz - 12\lambda$

$$\Rightarrow \left.\begin{aligned} \frac{\partial F}{\partial x} &= 3y + 2z + \lambda yz = 0 \\ \frac{\partial F}{\partial y} &= 3x + 2z + \lambda xz = 0 \\ \frac{\partial F}{\partial z} &= 2x + 2y + \lambda xy = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda &= \frac{-3}{z} + \frac{-2}{y} \\ \lambda &= \frac{-3}{z} + \frac{-2}{x} \\ \lambda &= \frac{-2}{y} + \frac{-2}{x} \end{aligned}$$

$$\frac{\partial F}{\partial \lambda} = xyz - 12 = 0$$

$$\Rightarrow \frac{-3}{z} + \frac{-2}{y} = \frac{-3}{z} + \frac{-2}{x} \Rightarrow \frac{-2}{y} = \frac{-2}{x} \Rightarrow x = y$$

$$\frac{-3}{z} + \frac{-2}{x} = \frac{-2}{y} + \frac{-2}{x} \Rightarrow \frac{-3}{z} = \frac{-2}{y} \Rightarrow z = \frac{3}{2} y$$

$$xyz - 12 = 0 \Rightarrow \frac{3}{2} y^3 - 12 \Rightarrow y^3 = 8 \Rightarrow y = 2$$

$$\Rightarrow x = 2 \Rightarrow z = 3.$$

$\Rightarrow$ (Assuming min exists) min is when $x = 2, y = 2, z = 3$

Q25

$$F(x, y, \lambda) = f(x, y) + \lambda a x + \lambda b y - \lambda c$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda a \quad , \quad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda b$$

At optimal $\frac{\partial f}{\partial x} + \lambda a = 0 = \frac{\partial f}{\partial y} + \lambda b$

$$\Rightarrow \lambda = \frac{-\frac{\partial f}{\partial x}}{a} = \frac{-\frac{\partial f}{\partial y}}{b} \Rightarrow \frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{a}{b}$$

Something useful to know: If $F(x)$ is antiderivative of $f(x)$

$\Rightarrow$ $\frac{1}{a}F(ax+b)$ is an antiderivative of $f(ax+b)$.

(a, b constant). E.g. $\frac{1}{2}e^{2x+3}$ is antiderivative of e^{2x+3}.

§7.6

Q1/

$$\int_0^1 e^{x+y}\,dy = e^{x+y}\Big|_0^1 = e^{x+1} - e^{x}$$

$$\Rightarrow \int_0^1 \left(\int_0^1 e^{x+y}\,dy\right)dx = \int_0^1 e^{x+1} - e^{x}\,dx = e^{x+1} - e^{x}\Big|_0^1$$

$$= (e^2 - e) - (e - 1) = e^2 - 2e + 1$$

Q2/

$$\int_{-1}^1 xy\,dx = \frac{x^2}{2}y\Big|_{-1}^1 = \frac{1^2}{2}y - \frac{(-1)^2}{2}y = 0$$

$$\Rightarrow \int_{-1}^1 \left(\int_{-1}^1 xy\,dx\right)dy = \int_{-1}^1 0\,dy = 1\Big|_{-1}^1 = 1-1 = 0$$

Q3/ $\int_{-1}^{1} x e^{xy}\, dy = \frac{x}{x} e^{xy} \Big|_{-1}^{1} = e^{x} - e^{-x}$

$\Rightarrow \int_{-2}^{0} \left(\int_{-1}^{1} x e^{xy}\, dy \right) dx = e^{x} + e^{-x} \Big|_{-2}^{0} = 2 - (e^{-2} + e^{2})$

Q4/ $\int_{-1}^{1} \frac{1}{3} y^{3} x\, dy = \frac{1}{12} y^{4} x \Big|_{-1}^{1} = 0$

$\Rightarrow \int_{0}^{1} \left(\int_{-1}^{1} \frac{1}{3} y^{3} x\, dy \right) dx = \int_{0}^{1} 0\, dx = 0.$

Q4/ $\iint_{R} x y^{2}\, dx\, dy = \int_{2}^{3} \left(\int_{0}^{2} x y^{2}\, dx \right) dy = \int_{2}^{3} \frac{x^{2} y^{2}}{2} \Big|_{0}^{2} dy$

$= \int_{2}^{3} 2 y^{2}\, dy = \frac{2}{3} y^{3} \Big|_{2}^{3} = 18 - \frac{16}{3} = \frac{38}{3}$

Q12/ $\int_0^2 e^{y-x}\,dx = -e^{y-x}\Big|_0^2 = (-e^{y-2}) - (-e^{y}) = e^{y} - e^{y-2}$

$\Rightarrow \iint_R e^{y-x}\,dx\,dy = \int_2^3 e^{y} - e^{y-2}\,dy = e^{y} - e^{y-2}\Big|_2^3 = (e^3 - e) - (e^2 - 1)$

$= e^3 - e^2 - e + 1.$