HW 12 Solutions

§ 12.4

1/ $k = 3 \Rightarrow \mathbb{E}(X) = \frac{1}{3}$ and $Var(X) = \frac{1}{3^2}$

6/ $\mathbb{E}(X) = \frac{1}{2} \Rightarrow k = 2 \Rightarrow f(x) = 2e^{-2x}$

$$Pr(X \le \tfrac{1}{3}) = \int_0^{1/3} 2e^{-2x}\,dx = -e^{-2x}\Big|_0^{1/3} = 1 - e^{-2/3}$$

10/ $\mathbb{E}(X) = 20 \Rightarrow k = \frac{1}{20} \Rightarrow f(x) = \frac{1}{20}e^{-\frac{1}{20}x}$

$$\Rightarrow Pr(10 \le X \le 30) = \int_{10}^{30} \frac{1}{20} e^{-\frac{1}{20}x}\,dx = -e^{-\frac{1}{20}x}\Big|_{10}^{30}$$

$$= e^{-\frac{1}{2}} - e^{-\frac{3}{2}}.$$

11/ $\mathbb{E}(X) = 2 \Rightarrow k = \frac{1}{2} \Rightarrow f(x) = \frac{1}{2}e^{-\frac{1}{2}x}$

$$Pr(X \le 4) = \int_0^4 \frac{1}{2} e^{-\frac{1}{2}x}\,dx = -e^{-\frac{1}{2}x}\Big|_0^4 = 1 - e^{-2}$$

15/ $\mu = \mathbb{E}(X) = 4$

σ = Standard Deviation of $X = \sqrt{Var(X)} = 1$

18/ $\mathbb{E}(X) = 3$

$Var(X) = 25$

19/ $f'(x) = -x e^{-\frac{x^2}{2}} \Rightarrow f'(0) = 0 \Rightarrow x = 0$ is critical number

$f''(x) = -e^{-\frac{x^2}{2}} + x^2 e^{-\frac{x^2}{2}} = (x^2 - 1) e^{-\frac{x^2}{2}} \Rightarrow f''(0) = -1 < 0$

$\Rightarrow f(x)$ has relative max at $x = 0$.

24/ a) $A(1.5) - A(0.5)$

b) $2A(0.75)$

c) $\frac{1}{2} - A(0.3)$

d) $\frac{1}{2} + A(1)$

Check table for exact values.

29/ X_1 = time to get to class with Capital Beltway

X_2 = time to get to class with local roads.

$$\Pr(X_1 \leq 30) = \Pr\left(Z \leq \frac{30-25}{5}\right) = \Pr(Z \leq 1)$$

Probability of getting to class on time

Standard Normal

$$\Pr(X_2 \leq 30) = \Pr\left(Z \leq \frac{30-28}{3}\right) = \Pr\left(Z \leq \frac{2}{3}\right)$$

$\Pr(Z \leq \frac{2}{3}) < \Pr(Z \leq 1) \Rightarrow$ She should take ~~local roads~~ Capital Beltway.

Q32/ $\Pr(500 \leq X \leq 600) = \Pr\left(\frac{500-525}{100} \leq Z \leq \frac{600-525}{100}\right)$

$$= \Pr(-0.35 \leq Z \leq 0.75) = A(0.75) + A(0.35)$$

34/ $\displaystyle\int_0^M ke^{-kx}\,dx = \frac{1}{2} \Rightarrow -e^{-kx}\Big|_0^M = \frac{1}{2}$

$$\Rightarrow 1 - e^{-kM} = \frac{1}{2} \Rightarrow e^{-kM} = \frac{1}{2} \Rightarrow e^{kM} = 2$$

$$\Rightarrow kM = \ln(2) \Rightarrow M = \ln(2)/k.$$

§12.5

5/ a) $Pr(X=0) = \frac{10^0}{0!} e^{-10} = e^{-10}$

1 by convention

b) $Pr(X \leq 2) = Pr(X=0) + Pr(X=1) + Pr(X=2)$

$= e^{-10} + \frac{10}{1!} e^{-10} + \frac{10^2}{2!} e^{-10}$

c) $Pr(X \geq 3) = 1 - Pr(X \leq 2)$

$= 1 - e^{-10} - \frac{10}{1!} e^{-10} - \frac{10^2}{2!} e^{-10}$

7/ a) $Pr(X=0) = e^{-\frac{3}{2}}$

b) $Pr(X = 2 \text{ or } 3) = Pr(X=2) + Pr(X=3)$

$= \frac{(\frac{3}{2})^2}{2!} e^{-\frac{3}{2}} + \frac{(\frac{3}{2})^3}{3!} e^{-3/2}$

c) $Pr(X \geq 4) = 1 - Pr(X \leq 3)$

$= 1 - e^{-3/2} - \frac{(\frac{3}{2})^2}{2!} e^{-3/2} - \frac{(\frac{3}{2})}{1!} e^{-3/2} - \frac{(\frac{3}{2})^3}{3!} e^{-3/2}$

12/

a) Probability of defective selection $= \frac{1}{40}$

$Pr(X=0) = \frac{1}{40}$

$Pr(X=1) = \underset{\text{1st passes}}{\frac{39}{40}} \cdot \underset{\text{2nd defective}}{\frac{1}{40}}$

$Pr(X=2) = \underset{\text{1st/2nd pass}}{\left(\frac{39}{40}\right)^2} \cdot \underset{\text{3rd defective}}{\frac{1}{40}}$

$\Rightarrow Pr(X=n) = \underset{\text{1st } n \text{ pass}}{\left(\frac{39}{40}\right)^n} \cdot \underset{n+1 \text{ fails}}{\frac{1}{40}}$ $\leftarrow$ geometric with $p = \frac{39}{40}$

b) $Pr(X=4) = \left(\frac{39}{40}\right)^4 \cdot \frac{1}{40}$

13/ a) X = number of Red before Blue

Probability that you see Red and not Blue $= \frac{3}{4}$

$\Pr(X = 0) = \frac{1}{4}$

$\Pr(X = 1) = \frac{3}{4} \cdot \frac{1}{4}$

1st Red 2nd Blue

$\Pr(X = 2) = \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$

1st/2nd Red 3rd Blue

$\Pr(X = n) = \left(\frac{3}{4}\right)^n \cdot \frac{1}{4}$ geometric with $p = \frac{3}{4}$

1st n Red n+1 Blue

$\Pr(X=3) + \Pr(X=4) + \dots$

b) $\Pr(X \geq 3) = \left(\frac{3}{4}\right)^3 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4} + \dots$

$$= \frac{\left(\frac{3}{4}\right)^3 \cdot \frac{1}{4}}{1 - \frac{3}{4}} = \left(\frac{3}{4}\right)^3$$

c) $E(X) = \frac{\frac{3}{4}}{1 - \frac{3}{4}} = 3.$

14/ X = number of students without cavity seen before one with cavity.

$\Pr(X=0) = \frac{2}{3}$

1st student has cavity

$\Pr(X=1) = \frac{1}{3} \cdot \frac{2}{3}$

1st cavity free

2nd has cavity

$\Pr(X=2) = \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3}$

1st/2nd cavity free

3rd has cavity

$\Pr(X=n) = \left(\frac{1}{3}\right)^n \cdot \frac{2}{3}$ ← geometric with $p = \frac{1}{3}$

$\Pr(X=2) = \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} = \frac{2}{27}$ ← probability 3rd student is 1st with cavity.

17/ Probability of random person showing symptoms $= \frac{5}{100} = \frac{1}{20}$

X = number with no symptoms seen before one with symptoms

$$Pr(X = 4) = \left(\frac{95}{100}\right)^4 \cdot \left(\frac{1}{20}\right)$$

1st 4 have no symptoms

5th shows symptoms

18/ X = number of tails before heads

a) $$Pr(X=n) = \left(\frac{1}{2}\right)^n \cdot \frac{1}{2}$$ ← geometric with $p = \frac{1}{2}$

1st n are tails

n+1 is head.

b) $$\mathbb{E}(X) = \frac{p}{1-p} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

c) $$Var(X) = (0-1)^2 Pr(X=0) + (1-1)^2 Pr(X=1) + (2-1)^2 Pr(X=2) + \dots$$

$$= \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} + 2^2 \cdot \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} + 3^2 \left(\frac{1}{2}\right)^4 \cdot \frac{1}{2} + 4^2 \cdot \left(\frac{1}{2}\right)^5 \cdot \frac{1}{2} + \dots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^3 \left(1 + 4 \cdot \left(\frac{1}{2}\right) + 9 \cdot \left(\frac{1}{2}\right)^2 + 16 \left(\frac{1}{2}\right)^3 + \dots\right)$$

Back to 28 from 11.5

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots$$ ← differentiate

$$\Rightarrow \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \dots$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = 1 + 4x + 9x^2 + 16x^3 + \dots$$

$$= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

$$\Rightarrow Var(X) = \frac{1}{2} + \left(\frac{1}{2}\right)^3 \cdot \left(\frac{(1-\frac{1}{2})^2 + 2\cdot(\frac{1}{2})(1-\frac{1}{2})}{(1-\frac{1}{2})^4} \right)$$

$$= \frac{1}{2} + \frac{12}{8} = 2$$