HW 11 Solutions

§12.1

2/ $\mathbb{E}(X) = 1 \cdot \frac{4}{9} + 2 \cdot \frac{4}{9} + 3 \cdot \frac{1}{9} = \frac{4+8+3}{9} = \frac{5}{3}$

$\mathrm{Var}(X) = (1 - \frac{5}{3})^2 \cdot \frac{4}{9} + (2 - \frac{5}{3})^2 \cdot \frac{4}{9} + (3 - \frac{5}{3})^2 \cdot \frac{1}{9}$

5/ a)

Outcome	0	1	2	3
Probability	$\frac{11}{52}$	$\frac{26}{52}$	$\frac{13}{52}$	$\frac{2}{52}$

b) $\mathbb{E}(X) = 0 \cdot \frac{11}{52} + 1 \cdot \frac{26}{52} + 2 \cdot \frac{13}{52} + 3 \cdot \frac{2}{52}$

$= \frac{58}{52}$

c) On average there are $\frac{58}{52}$ accidents each week.

4/ X = profit with no protection

Outcome	100,000	60,000
Probability	0.75	0.25

$\Rightarrow \mathbb{E}(X) = 0.75 \cdot (100,000) + 0.25 \cdot (60,000$

$= 90,000$

$90,000 < 100,000 - 5,000 \Rightarrow$ Should pay \$5000 to protect crops.

§12.2

5/ 1/ $5x^4 \geq 0$ on $[0,1]$

2/ $\int_0^1 5x^4 \, dx = x^5 \Big|_0^1 = 1$

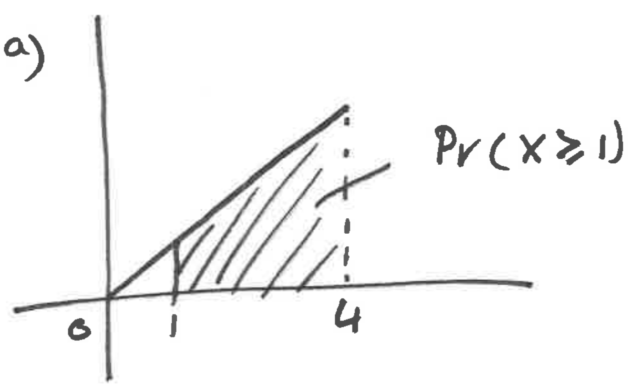
9/ 1/ $f(x) = k \geqslant 0$ on $[5, 20] \Rightarrow k \geqslant 0$

2/ $\int_5^{20} k\,dx = kx \Big|_5^{20} = 15k = 1 \Rightarrow k = \frac{1}{15}$

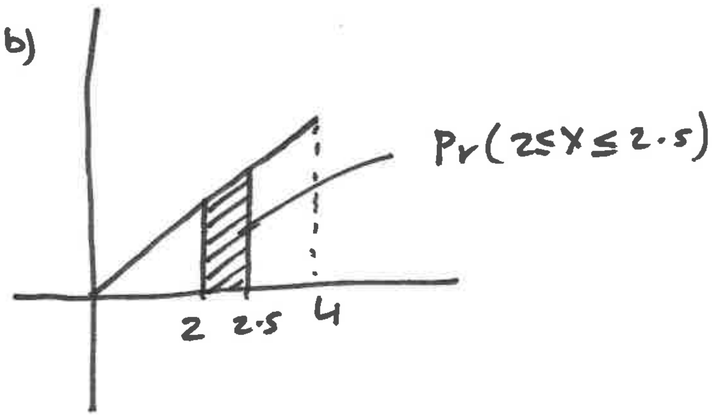
11/ 1/ $f(x) \geqslant 0$ on $[0,1] \Rightarrow kx^2(1-x) \geqslant 0$ on $[0,1] \Rightarrow k \geqslant 0$

2/ $\int_0^1 kx^2(1-x)\,dx = \frac{k}{3}x^3 - \frac{k}{4}x^4 \Big|_0^1 = k\left(\frac{1}{3} - \frac{1}{4}\right) = 1 \Rightarrow k = 12.$

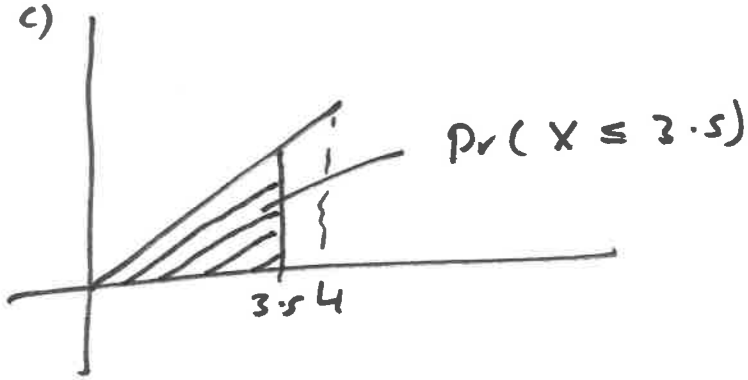
13/ a)



b)



c)



19/ $\Pr(X \geqslant 35) = \int_{35}^{50} \frac{1}{20}\,dx = \frac{1}{20}x\Big|_{35}^{50} = \frac{15}{20} = \frac{3}{4}$

20/ $\Pr(0 \leq X \leq 4) = \int_{0}^{4} \frac{11}{10(x+1)^2}\,dx = \frac{-11}{10(x+1)}\Big|_{0}^{4} = \frac{11}{10} - \frac{11}{50} = \frac{44}{50}$

21/ $\underset{\text{C.D.F.}}{F(x)} = \frac{1}{2}\sqrt{x-1} \Rightarrow \underset{\text{P.D.F}}{f(x)} = \frac{1}{4}(x-1)^{-\frac{1}{2}}$

24/ $F'(x) = \frac{1}{2}(3-x)$ <u>and</u> $F(1) = 0$

$\Rightarrow F(x) = C + \frac{3}{2}x - \frac{1}{4}x^2$ and $0 = C + \frac{3}{2} - \frac{1}{4} \Rightarrow C = \frac{-5}{4}$

$\Rightarrow F(x) = \frac{3}{2}x - \frac{1}{4}x^2 - \frac{5}{4}$

26/ a) $\Pr(3 \leq X \leq 4) = \int_{3}^{4} \frac{4}{9}x - \frac{1}{9}x^2\,dx = \frac{2}{9}x^2 - \frac{1}{27}x^3\Big|_{3}^{4}$

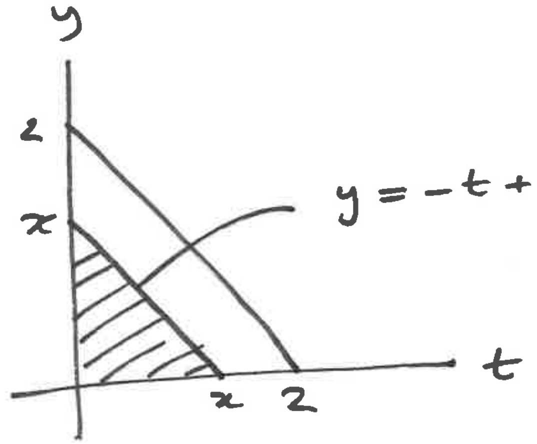
$= \frac{32}{9} - \frac{64}{27} - 2 + 1$

b) $F'(x) = \frac{4}{9}x - \frac{1}{9}x^2$ <u>and</u> $F(1) = 0$

$\Rightarrow F(x) = C + \frac{2}{9}x^2 - \frac{x^3}{27}$ and $0 = C + \frac{2}{9} - \frac{1}{27} \Rightarrow C = \frac{-5}{27}$

$\Rightarrow F(x) = \frac{-5}{27} + \frac{2}{9}x^2 - \frac{x^3}{27}$

29/



Sum of coordinates

$\downarrow$

$y = -t + x \Rightarrow y + t = x$

$\Rightarrow X \leq x$ iff and only iff P, the selected point, is in shaded region $\Rightarrow \Pr(X \leq x) = \dfrac{\text{Area}(\equiv)}{\text{Total area}} = \dfrac{\frac{x^2}{2}}{\frac{2^2}{2}} = \dfrac{x^2}{4}$

$\parallel$

$F(x)$ (C.D.F.)

30/ $\underset{\text{C.D.F.}}{F(x)} = \dfrac{x^2}{4} \Rightarrow \underset{\text{P.D.F}}{f(x)} = \dfrac{x}{2}$

37/ a) 1/ $4x^{-5} \geqslant 0$ on $[1, \infty)$

2/ $\displaystyle\int_1^\infty 4x^{-5}\,dx = \lim_{t\to\infty} -x^{-4}\Big|_1^t = \lim_{t\to\infty}\left(1 - \frac{1}{t^4}\right) = 1$ $\Big\}$ P.D.F. on $[1, \infty)$

b) $F'(x) = 4x^{-5}$, $F(1) = 0 \Rightarrow F(x) = C - x^{-4}$ and $0 = C - 1$

$\Rightarrow C = 1 \Rightarrow F(x) = 1 - x^{-4}$

c) $Pr(1 \le X \le 2) = F(2) - F(1) = (1-2^{-4}) - (1-1^{-4})$

$= 1 - \frac{1}{16} = \frac{15}{16}$

$Pr(X \geq 2) = 1 - Pr(X < 2) = 1 - \frac{15}{16} = \frac{1}{16}$

§12.3

7/ $\mathbb{E}(X) = \int_0^1 x \cdot 12x(1-x)^2\,dx = \int_0^1 12x^2 - 24x^3 + 12x^4\,dx$

$= 4x^3 - 6x^4 + \frac{12}{5}x^5 \Big|_0^1 = 4 - 6 + \frac{12}{5} = \frac{2}{5}$

$Var(X) = \int_0^1 x^2 \cdot 12x(1-x)^2\,dx - \left(\frac{2}{5}\right)^2$

$= \int_0^1 12x^3 - 24x^4 + 12x^5\,dx - \frac{4}{25}$

$= 3x^4 - \frac{24}{5}x^5 + 2x^6 \Big|_0^1 - \frac{4}{25} = 3 - \frac{24}{5} + 2 - \frac{4}{25}$

9/ a) $f(x) = 30x^2(1-x)^2 = 30x^2 - 60x^3 + 30x^4$

$\Rightarrow F(x) = C + 16x^3 - 15x^4 + 6x^5$ and $F(0) = 0 \Rightarrow C = 0$

$\Rightarrow F(x) = 10x^3 - 15x^4 + 6x^5$

b) $\Pr(X \leq 0.25) = F(0.25) - F(0) = 10 \cdot (0.25)^3 - 15(0.25)^4 + 6(0.25)^5$

c) $\mathbb{E}(X) = \int_0^1 x \cdot 30x^2(1-x)^2 dx = \int_0^1 30x^3 - 60x^4 + 30x^5 \, dx$

$= \frac{30}{4}x^4 - \frac{60}{5}x^5 + \frac{30}{6}x^6 \Big|_0^1 = \frac{30}{4} - 12 + 5 = \frac{2}{4} = \frac{1}{2}$

On average 50% of paper is adverts

d) $Var(X) = \int_0^1 x^2 \cdot 30x^2(1-x)^2 dx - \frac{1}{4}$

$= \int_0^1 30x^4 - 60x^5 + 30x^6 \, dx - \frac{1}{4}$

$= 6x^5 - 10x^6 + \frac{30}{7}x^7 \Big|_0^1 - \frac{1}{4} = 6 - 10 + \frac{30}{7} - \frac{1}{4}$

11/ $F(x) = \frac{1}{9}x^2$ on $[0,3] \Rightarrow f(x) = \frac{2}{9}x$ on $[0,3]$

a) $\mathbb{E}(X) = \int_0^3 x \cdot \frac{2}{9}x \, dx = \frac{2}{27}x^3 \Big|_0^3 = 2 \Leftarrow$ average assembly time

b) $Var(X) = \int_0^3 x^2 \cdot \frac{2}{9}x \, dx - 4 = \frac{1}{18}x^4 \Big/_0^3 - 4 = \frac{81}{18} - 4$

13/ $\mathbb{E}(X) = \int_0^{12} x \cdot \frac{1}{72} x \, dx = \frac{1}{72} \frac{x^3}{3} \Big|_0^{12} = \frac{1}{72} \frac{12^3}{3} = \frac{2}{3} \cdot 12 = 8$

average time spent reading editorial page.

15/ a) $f(x) = \frac{1}{3}x - \frac{x^2}{18}$

$F(x) = C + \frac{1}{6}x^2 - \frac{x^3}{54}$ and $F(3) = 0$

$\Rightarrow 0 = C + \frac{9}{6} - \frac{27}{54} \Rightarrow 0 = C + 1 \Rightarrow C = -1$

$\Rightarrow F(x) = \frac{1}{6}x^2 - \frac{x^3}{54} - 1$

b) $Pr(3 \le X \le 5) = F(5) - F(3) = \frac{5^2}{6} - \frac{5^3}{54} - 1$

c) $\mathbb{E}(X) = \int_3^6 x \cdot \left(\frac{1}{3}x - \frac{x^2}{18}\right) dx = \frac{1}{9}x^3 - \frac{1}{4 \cdot 18}x^4 \Big|_3^6 = 4.125 \Rightarrow$ 412.5 hours worked on average.

d) $Var(X) = \int_3^6 \frac{1}{3}x^3 - \frac{x^4}{18} dx - (4.125)^2$

$= \frac{1}{12}x^4 - \frac{1}{18} \cdot \frac{x^5}{5} \Big|_3^6 - (4.125)^2 = 0.5344$

14/ $\displaystyle\int_0^M \frac{1}{18} x \, dx = \frac{1}{2} \Rightarrow \frac{1}{36} x^2 \Big|_0^M = \frac{1}{2}$

$\Rightarrow \dfrac{M^2}{36} = 1/2 \Rightarrow M^2 = 18 \Rightarrow M = \sqrt{18}$

21/ $f(x) = \dfrac{1}{9} x^2$ $\quad \displaystyle\int_0^M \frac{1}{9} x^2 \, dx = \frac{1}{2} \Rightarrow \frac{1}{27} x^3 \Big|_0^M = \frac{1}{2}$

$\Rightarrow \dfrac{1}{27} M^3 = \dfrac{1}{2} \Rightarrow M = \dfrac{3}{\sqrt[3]{2}}$

A machine component has 50% chance of lasting at least $\dfrac{3}{\sqrt[3]{2}}$ hundred hours.