HW 10 Solutions

§11.4

1/ Let $f(x) = \frac{3}{\sqrt{x}}$ $\Rightarrow$ 1/ $f(k) = \frac{3}{\sqrt{k}}$ for $k \geq 1$

2/ $f'(x) = \frac{-\frac{3}{2}}{x^{3/2}} < 0$ on $[1, \infty)$

$\Rightarrow f(x)$ decreasing on $[1, \infty)$

Can apply Integral Test

$$\int \frac{3}{\sqrt{x}}\,dx = 6\sqrt{x} + C \Rightarrow \int_1^t \frac{3}{\sqrt{x}}\,dx = 6\sqrt{x}\Big|_1^t = 6\sqrt{t} - 6$$

$$\Rightarrow \int_1^\infty \frac{3}{\sqrt{x}}\,dx = \lim_{t\to\infty} 6\sqrt{t} - 6 = \infty \Rightarrow \sum_{k=1}^{\infty} \frac{3}{\sqrt{k}} \text{ divergent}$$

3/ $f(x) = \frac{1}{(x-1)^3}$ $\Rightarrow$ 1/ $f(k) = \frac{1}{(k-1)^3}$ for $k \geq 2$

2/ $f'(x) = \frac{-3}{(x-1)^4} < 0$ on $[2, \infty)$

$\Rightarrow f(x)$ decreasing on $[2, \infty)$

$$\int \frac{1}{(x-1)^3}\,dx = \frac{-1}{2(x-1)^2} + C \Rightarrow \int_2^t \frac{1}{(x-1)^3}\,dx = \frac{-1}{2(x-1)^2}\Big|_2^t$$

$$= \frac{1}{2} - \frac{1}{2(t-1)^2} \Rightarrow \int_2^\infty \frac{1}{(x-1)^3}\,dx = \lim_{t\to\infty} \frac{1}{2} - \frac{1}{2(t-1)^2} = \frac{1}{2}$$

$\Rightarrow$ convergent $\Rightarrow \sum_{k=2}^{\infty} \frac{1}{(k-1)^3}$ convergent.

6/ $f(x) = \dfrac{1}{x\sqrt{\ln(x)}} \Rightarrow$ 1/ $f(k) = \dfrac{1}{k\sqrt{\ln(k)}}$ for $k \geq 2$

2/ $$f'(x) = \frac{-\left(\sqrt{\ln(x)} + x \cdot \frac{1}{x} \cdot \frac{1}{2}(\ln(x))^{-\frac{1}{2}}\right)}{\left(x\sqrt{\ln(x)}\right)^2} < 0$$

on $[2, \infty)$

$\Rightarrow f(x)$ decreasing on $[2, \infty)$

Let $u = \ln(x) \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow dx = x\,du \Rightarrow$

$$\int \frac{1}{x\sqrt{\ln(x)}}\,dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C = 2\sqrt{\ln(x)} + C$$

$$\Rightarrow \int_2^t \frac{1}{x\sqrt{\ln(x)}}\,dx = 2\sqrt{\ln(x)}\Big|_2^t = 2\sqrt{\ln(t)} - 2\sqrt{\ln(2)}$$

$$\Rightarrow \int_2^\infty \frac{1}{x\sqrt{\ln(x)}}\,dx = \lim_{t \to \infty} 2\sqrt{\ln(t)} - 2\sqrt{\ln(2)} = \infty \Rightarrow \text{divergent}$$

$\Rightarrow \displaystyle\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln(k)}}$ divergent.

7/ $f(x) = \dfrac{x}{(x^2+1)^{3/2}} \Rightarrow$ 1/ $f(k) = \dfrac{k}{(k^2+1)^{3/2}}$ for $k \geq 2$

2/ $$f'(x) = \frac{(x^2+1)^{3/2} - x \cdot 2x \cdot \frac{3}{2}(x^2+1)^{\frac{1}{2}}}{(x^2+1)^3}$$

$$\Rightarrow \quad f'(x) = \frac{(x^2+1)^{\frac{1}{2}}\left((x^2+1) - 3x^2\right)}{(x^2+1)^3} = \frac{1-2x^2}{(x^2+1)^{5/2}}$$

$$\Rightarrow \quad f'(x) < 0 \iff 1-2x^2 < 0 \iff x^2 > \frac{1}{2}$$

$\Rightarrow$ $f(x)$ decreasing on $\left[\frac{1}{\sqrt{2}}, \infty\right)$ $\Rightarrow$ Can apply integral test.

Let $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow$

$$\int \frac{x}{(x^2+1)^{3/2}}\,dx = \int \frac{1}{2u^{3/2}}\,du = \frac{-1}{\sqrt{u}} + C = \frac{-1}{\sqrt{x^2+1}} + C$$

$$\Rightarrow \int_2^t \frac{x}{(x^2+1)^{3/2}}\,dx = \frac{-1}{\sqrt{x^2+1}}\Bigg|_2^t = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{t^2+1}}$$

$$\Rightarrow \int_2^\infty \frac{x}{(x^2+1)^{3/2}}\,dx = \lim_{t\to\infty} \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{t^2+1}} = \frac{1}{\sqrt{5}} \Rightarrow \text{convergent}$$

$\Rightarrow$ $\displaystyle\sum_{k=2}^{\infty} \frac{k}{(k^2+1)^{3/2}}$ convergent

9/ $f(x) = \dfrac{1}{x(\ln(x))^2} \Rightarrow$ 1/ $f(k) = \dfrac{1}{k(\ln(k))^2}$ for $k \geq 2$

2/ $f'(x) = \dfrac{-((\ln(x))^2 + 2\ln(x))}{(x(\ln(x))^2)^2} < 0$ on $[2, \infty)$

$\Rightarrow f(x)$ decreasing on $[2, \infty)$

Let $u = \ln(x) \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow dx = x\,du \Rightarrow$

$$\int \frac{1}{x(\ln(x))^2}\,dx = \int \frac{1}{u^2}\,du = \frac{-1}{u} + C = \frac{-1}{\ln(x)} + C$$

$$\Rightarrow \int_2^t \frac{1}{x(\ln(x))^2}\,dx = \frac{-1}{\ln(x)}\Big|_2^t = \frac{1}{\ln(2)} - \frac{1}{\ln(t)} \Rightarrow$$

$$\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx = \lim_{t \to \infty} \frac{1}{\ln(2)} - \frac{1}{\ln(t)} = \frac{1}{\ln(2)} \Rightarrow \text{convergent}$$

$$\Rightarrow \sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2} \quad \text{convergent.}$$

Q10/ $f(x) = \dfrac{x+1}{(x^2+2x+1)^2} \Rightarrow$ 1/ $f(k) = \dfrac{k+1}{(k^2+2k+1)^2}$ for $k \geq 2$

2/ $f'(x) = \dfrac{(x^2+2x+1)^2 - (x+1)2(2x+2)(x^2+2x+1)}{(x^2+2x+1)^4}$

$$\Rightarrow \quad f'(x) = \frac{(x^2+2x+1) - 4(x+1)^2}{(x^2+2x+1)^3} = \frac{-3(x+1)^2}{(x^2+2x+1)^3} < 0 \quad \text{on } [2, \infty)$$

$\Rightarrow$ $f(x)$ decreasing on $[2, \infty)$

Let $u = x^2 + 2x + 1 \Rightarrow \frac{du}{dx} = 2x + 2 \Rightarrow dx = \frac{du}{2x+2}$

$$\Rightarrow \int \frac{x+1}{(x^2+2x+1)^2}\,dx = \int \frac{1}{2} \cdot \frac{1}{u^2}\,du = \frac{-1}{2u} + C = \frac{-1}{2(x^2+2x+1)} + C$$

$$\Rightarrow \int_2^t \frac{x+1}{(x^2+2x+1)^2}\,dx = \frac{-1}{2(x^2+2x+1)}\Bigg|_2^t = \frac{1}{2(2^2+2\cdot 2+1)} - \frac{1}{2(t^2+2t+1)}$$

$$\Rightarrow \int_2^\infty \frac{x+1}{(x^2+2x+1)^2}\,dx = \lim_{t\to\infty}\left(\frac{1}{18} - \frac{1}{2(t^2+2t+1)}\right) = \frac{1}{18}$$

$\Rightarrow$ convergent $\Rightarrow \displaystyle\sum_{k=2}^{\infty} \frac{k+1}{(k^2+2k+1)^2}$ convergent.

13/ Let $c_k = \dfrac{3}{9+k^2}$ $\Rightarrow$ 1/ $f(k) = c_k$ for $k \geq 0$

$f(x) = \dfrac{3}{9+x^2}$ 2/ $f'(x) = \dfrac{-6x}{(9+x^2)^2} \leqslant 0$ on $[0, \infty)$

$\Rightarrow$ Can apply integral test.

$\displaystyle\int_0^\infty f(x)\,dx$ convergent $\Rightarrow$ $\displaystyle\sum_{k=0}^{\infty} \frac{3}{9+k^2}$ convergent.

14/ $f(x) = \dfrac{x}{e^x}$ $\Rightarrow$ 1/ $f(k) = \dfrac{k}{e^k}$ for $k \geqslant 1$

2/ $f'(x) = \dfrac{e^x - xe^x}{e^{2x}} = \dfrac{1-x}{e^x} \leqslant 0$ on $[1, \infty)$

$\Rightarrow$ $f(x)$ <u>decreasing</u> on $[1, \infty)$

$\displaystyle\int \underset{\substack{\uparrow\\ f(x)}}{x}\, \underset{\substack{\uparrow\\ g(x)}}{e^{-x}}\,dx \underset{\substack{\uparrow\\ \text{Parts}}}{=} -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$

$f'(x) = 1$, $G(x) = -e^{-x}$

$\Rightarrow \displaystyle\int_1^t x e^{-x}\,dx = -xe^{-x} - e^{-x} \Big|_1^t = \frac{-t}{e^t} - \frac{1}{e^t} + 2e^{-1}$

$\Rightarrow \displaystyle\int_1^\infty x e^{-x}\,dx = \lim_{t\to\infty} \frac{-t}{e^t} - \frac{1}{e^t} + 2e^{-1} = 2e^{-1}$ $\Rightarrow$ convergent

$$\Rightarrow \sum_{k=1}^{\infty} \frac{k}{e^k} \quad \text{convergent}$$

21/ $0 < \dfrac{1}{k^2+5} < \dfrac{1}{k^2}$ for $k \geq 2$

$$\sum_{k=2}^{\infty} \frac{1}{k^2} \text{ conv} \Rightarrow \sum_{k=2}^{\infty} \frac{1}{k^2+5} \text{ conv.}$$

p-series, $p=2$

22/ $0 < \dfrac{1}{k} = \dfrac{1}{\sqrt{k^2}} < \dfrac{1}{\sqrt{k^2-1}}$ for $k \geq 2$

$$\sum_{k=2}^{\infty} \frac{1}{k} \ \underline{\text{div}} \Rightarrow \sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2-1}} \ \underline{\text{div}}$$

p-series $p=1$

23/ $0 \leq \dfrac{1}{5^k} \cos^2\left(\dfrac{k\pi}{4}\right) \leq \dfrac{1}{5^k}$

$$\sum_{k=1}^{\infty} \frac{1}{5^k} \text{ conv} \Rightarrow \sum_{k=1}^{\infty} \frac{\cos^2\left(\frac{k\pi}{4}\right)}{5^k} \text{ conv.}$$

geo. $r = \dfrac{1}{5}$

27/ For $k \geq 3$ $\quad 0 < \dfrac{1}{k\ln(k)} \underset{\substack{\uparrow \\ \text{wrong way}}}{<} \dfrac{1}{k}$

Because $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ divergent we cannot use comparison test to determine $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ convergence / divergence.

28/ For $k \geq 3$ $\quad 0 < \dfrac{1}{k^2\ln(k)} < \dfrac{1}{k^2}$

$\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$ convergent $\Rightarrow$ $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2\ln(k)}$ convergent.

§11.5

1/ $f(x) = \dfrac{1}{2x+3}$ $\qquad f(0) = \dfrac{1}{3}$

$f'(x) = \dfrac{-2}{(2x+3)^2}$ $\qquad f'(0) = \dfrac{-2}{3^2}$

$f''(x) = \dfrac{2^2 \cdot 2}{(2x+3)^3}$ $\quad \Rightarrow \quad f''(0) = \dfrac{2^2}{3^3} \cdot 2$

$f'''(x) = \dfrac{2^3 \cdot 2 \cdot (-3)}{(2x+3)^4}$ $\qquad f'''(0) = \dfrac{2^3}{3^4} \cdot (-1)^3 \cdot 3!$

$f^{(4)}(x) = \dfrac{2^4 \cdot 2 \cdot (-3) \cdot (-4)}{(2x+3)^5}$ $\qquad f^{(4)}(0) = \dfrac{2^4}{3^5} \cdot (-1)^4 \cdot 4!$

$$\Rightarrow P(x) = \frac{1}{3} - \frac{2}{3^2}x + \frac{2^2}{3^3}x^2 - \frac{2^3}{3^4}x^3 + \frac{2^4}{3^5}x^4 \ldots$$

6/ $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$\Rightarrow \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 \ldots$

7/ $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 \ldots$

12/ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$\Rightarrow e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$

$\Rightarrow x^3 e^{x^2} = x^3 + x^5 + \frac{x^7}{2!} + \frac{x^9}{3!} + \frac{x^{11}}{4!} + \dots$

20/ $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots$

$\Rightarrow \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} \dots$

$\Rightarrow x\sin(x^2) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \frac{x^{19}}{9!} \dots$

23/ a) $f(x) = \frac{1}{2}(e^x + e^{-x})$

$\Rightarrow f'(x) = \frac{1}{2}(e^x - e^{-x}$

$\Rightarrow f''(x) = \frac{1}{2}(e^x + e^{-x})$

$\vdots$

$\Rightarrow$ $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$ $\Rightarrow p(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$\Rightarrow e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$\Rightarrow \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left(2 + 2\cdot\left(\frac{x^2}{2!}\right) + 2\cdot\left(\frac{x^4}{4!}\right) + 2\left(\frac{x^6}{6!}\right) + \dots\right)$

$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

25/ $$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \dots$$

$$\Rightarrow \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \dots$$

27/ $$\int \frac{1}{\sqrt{1+x^2}}\,dx = \int 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 \dots \,dx$$

$$= C + x - \frac{1}{3\cdot 2}x^3 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \dots$$

$$\Rightarrow \ln(x+\sqrt{1+x^2}) = C + x - \frac{1}{3\cdot 2}x^3 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} \dots$$

for some C

Evaluate at $x = 0$

$$\Rightarrow \ln(1) = 0 = C + 0 + 0 \dots \Rightarrow C = 0$$

$$\Rightarrow \ln(x+\sqrt{1+x^2}) = x - \frac{1}{3\cdot 2}x^3 + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \dots$$

33/ $\frac{f^{(4)}(0)}{4!}$ = coefficient in front of x^4 $= 0$

$$\Rightarrow f^{(4)}(0) = 0$$

35/ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \Rightarrow e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$

$$\Rightarrow \int e^{-x^2} dx = C + x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} \dots$$

38/ $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\Rightarrow \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

$$\Rightarrow \int \sin(x^2)\, dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \dots$$

$$\Rightarrow \int_0^1 \sin(x^2)\, dx = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \dots \Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \dots$$