<u>Trigonometric Functions</u>

Many natural phenomena repeat in set patterns. E.g. The tides, water waves, seasonal crop yields.

<u>Aim</u> : Introduce a new class of functions which are <u>periodic</u> (ie repeat themselves in set intervals).

How do we numerically quantify angles?

<u>Degrees</u> : Divide a full rotation into 360 equal pieces, each called a degree.

degrees

360° = full rotation

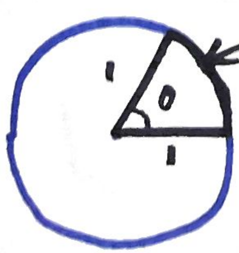
180° = half rotation

90° = quarter rotation

Problem : Very arbitrary. Why 360 degrees? There's no good reason

<u>Radians</u> :

circle radius 1



This length is the angle θ expressed in radians.

Recall : Circumference of a circle of radius 1 is 2π.

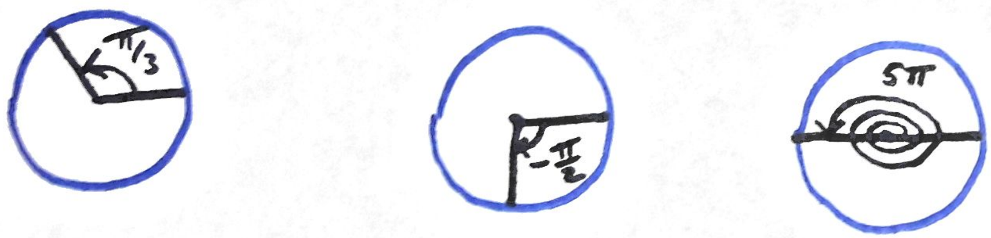
$\Rightarrow$ $360^\circ = 2\pi$ radians $\qquad 45^\circ = \pi/4$ radians

$180^\circ = \pi$ radians $\qquad 60^\circ = \pi/3$ radians

$90^\circ = \pi/2$ radians $\qquad 30^\circ = \pi/6$ radians

Benefit : Much less arbitrary. No dividing full rotation into pieces

Rotations in radians :



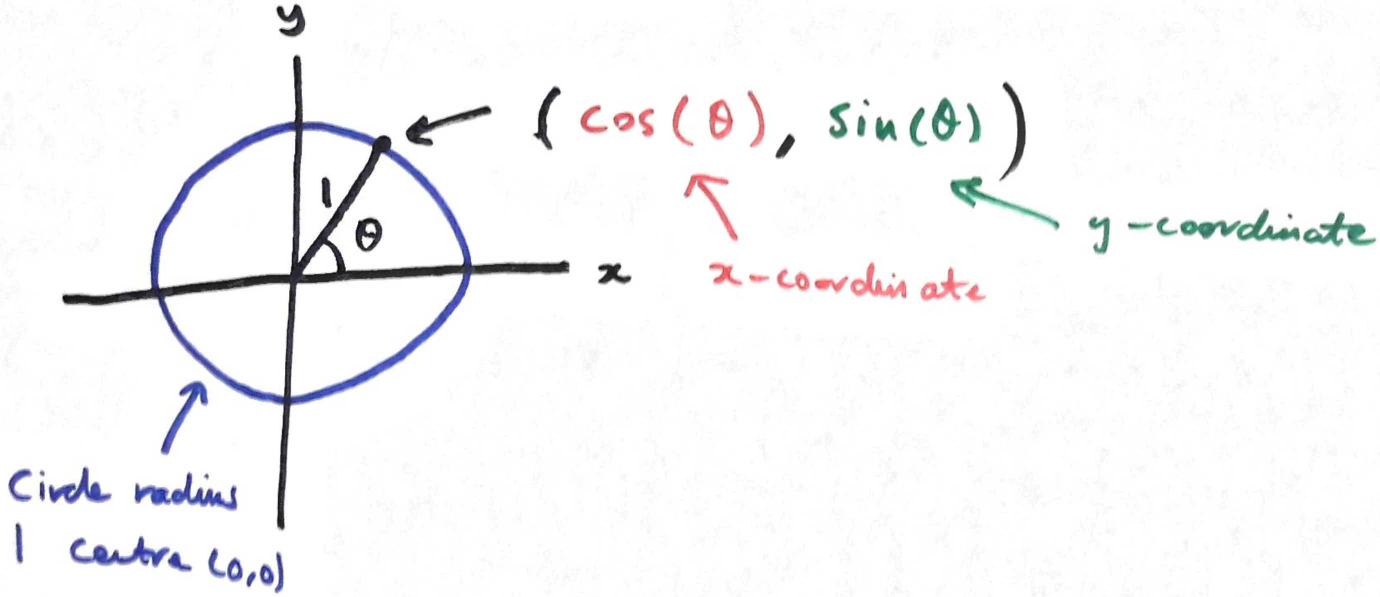
Convention : Anticlockwise is +
clockwise is −

Definition : Let θ be any number, thought of as a rotation in radians (about $(0,0)$ starting from positive x-axis)

We define $\sin(\theta)$ and $\cos(\theta)$ as follows :

$\sin(\theta)$: sine of θ

$\cos(\theta)$: cosine of θ



Remarks

1/ As θ varies the corresponding point on the circle varies. We can immediately see that

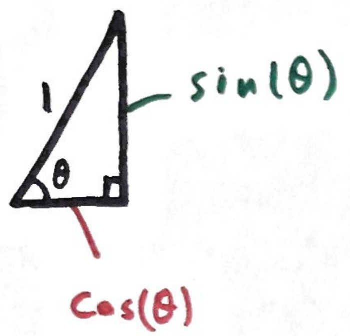
$$-1 \le \cos(\theta) \le 1 \quad \text{and} \quad -1 \le \sin(\theta) \le 1 \quad \text{for all } \theta$$

and

full rotation anticlockwise

$$\cos(\theta + 2\pi) = \cos(\theta) \quad \text{and} \quad \sin(\theta + 2\pi) = \sin(\theta)$$

1/



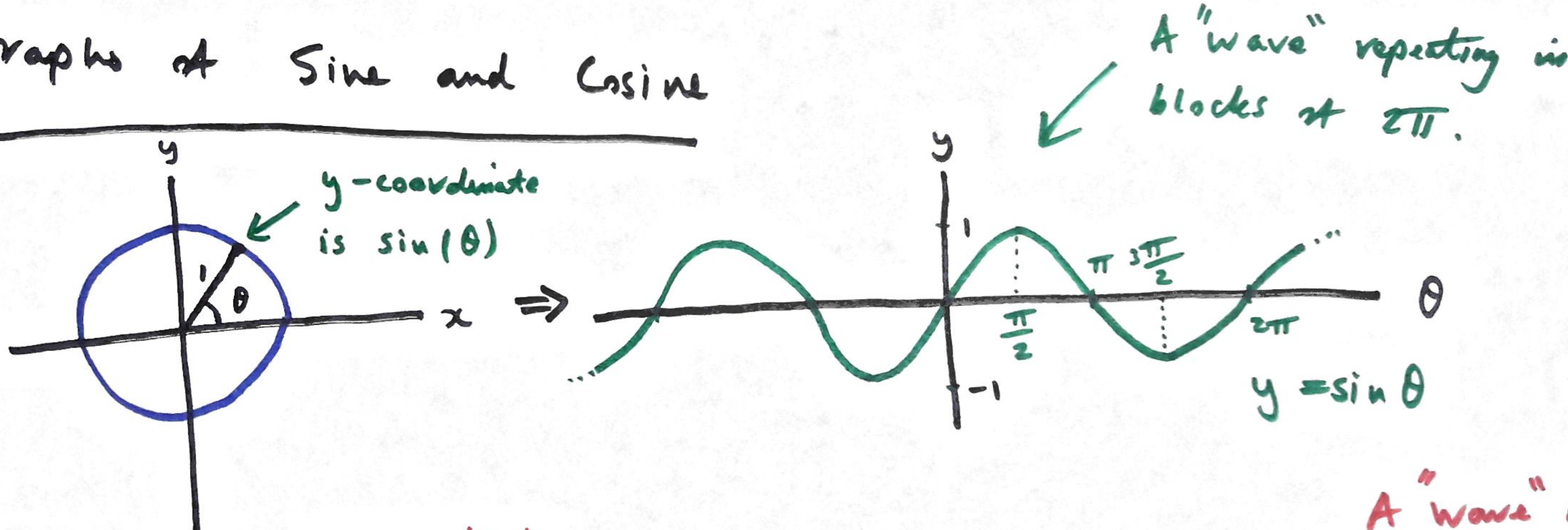
$\Longrightarrow$ (Pythagoras)

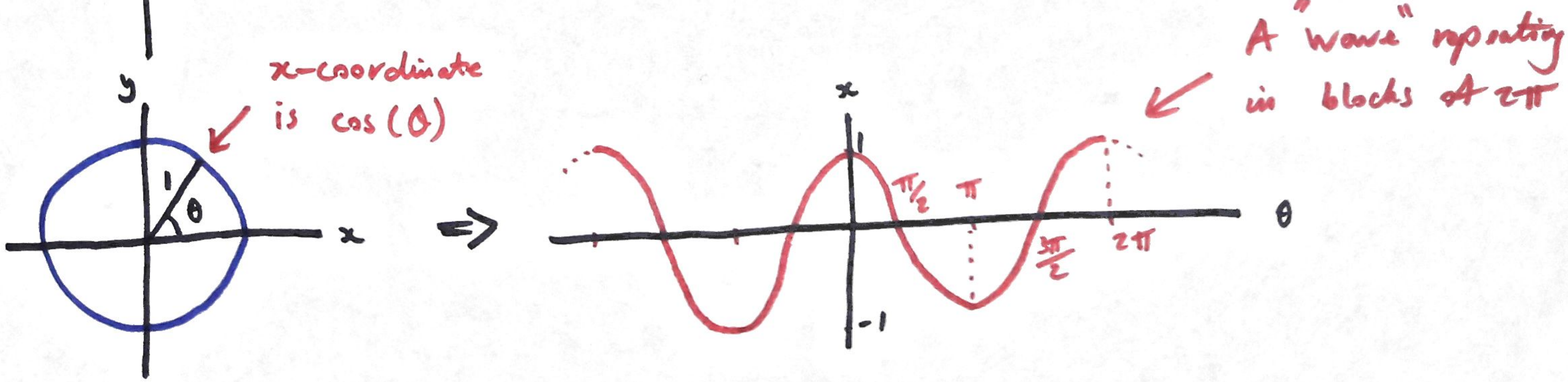
$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = (\sin\theta)^2, \qquad \cos^2\theta = (\cos\theta)^2$$

This is true for any θ.

Graphs of Sine and Cosine

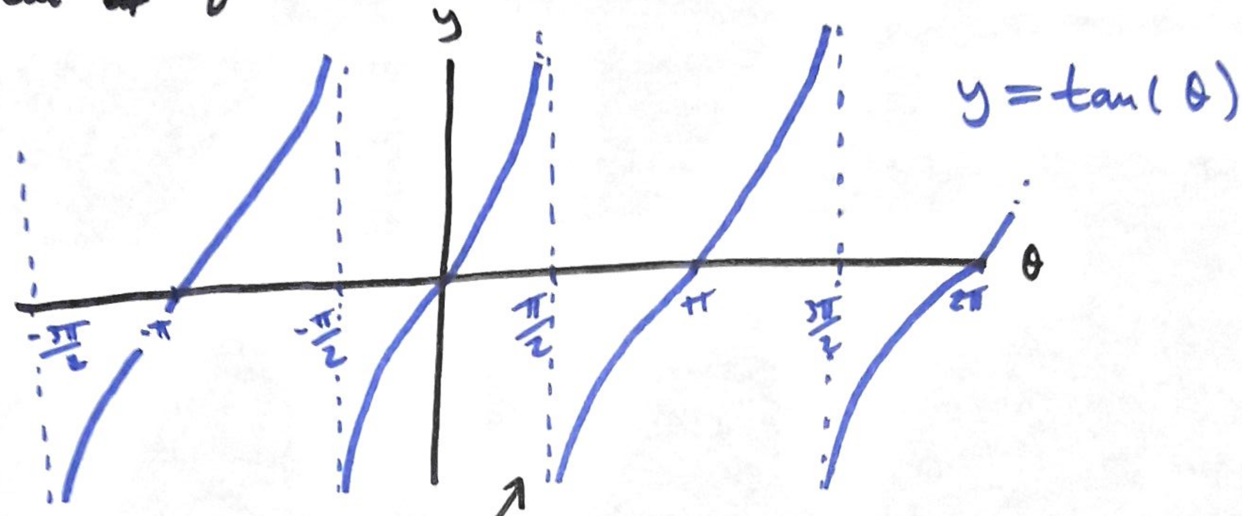




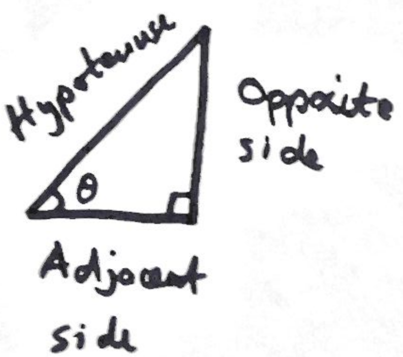
<u>Definition</u> $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$

tangent of θ

Graph :
(not obvious)



Geometric Interpretation of sin, cos and tan for $0 \leq \theta < \frac{\pi}{2}$



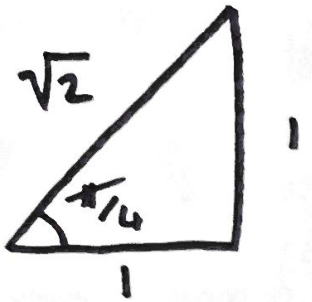
$\Longrightarrow$

$\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$

$\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$

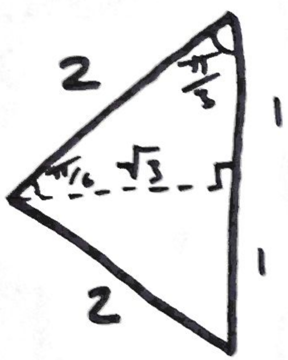
$\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}}$

Core Values :



$\Rightarrow$

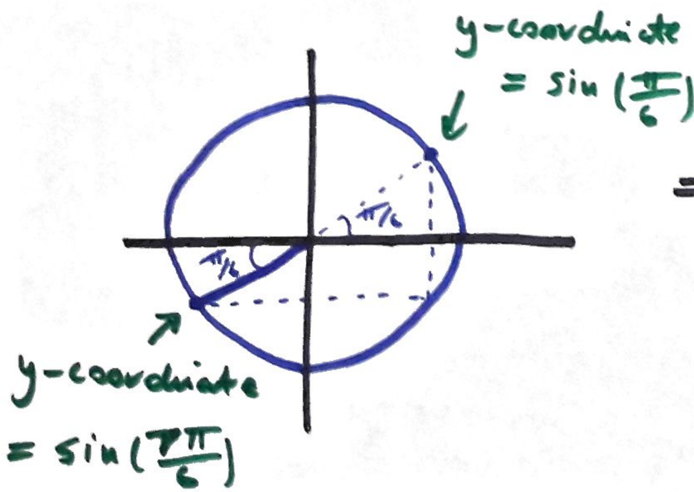
$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$\tan\left(\frac{\pi}{4}\right) = 1$$



$\Rightarrow$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

<u>Example</u> $\sin\left(\frac{7\pi}{6}\right) = ?$ $\quad \frac{7\pi}{6} = \pi + \frac{\pi}{6} \Rightarrow$



$$\Rightarrow \sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

Facts

$$\boxed{\begin{aligned} \frac{d}{d\theta}\sin(\theta) &= \cos(\theta) \\ \frac{d}{d\theta}\cos(\theta) &= -\sin(\theta) \end{aligned}}$$

These are <u>not</u> obvious. We will not justify them

$$\Rightarrow \quad \frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) \underset{\substack{\uparrow \\ \text{quotient} \\ \text{rule}}}{=} \frac{\cos\theta\cdot\cos\theta - (-\sin\theta)\sin\theta}{(\cos\theta)^2}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Definition : $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, $\cot(\theta) = \dfrac{1}{\tan(\theta)}$

(not so important for us)

Hence $\dfrac{d}{d\theta}\tan\theta = \sec^2\theta$

Integral versions :

$$\int \sin(\theta)\, d\theta = -\cos(\theta) + C$$

$$\int \cos(\theta)\, d\theta = \sin(\theta) + C$$

$$\int \sec^2(\theta)\, d\theta = \tan(\theta) + C$$

<u>Examples</u>

1/ $$\int_{\pi/6}^{\pi/4} \sin(\theta)\, d\theta = -\cos(\theta)\Big|_{\pi/6}^{\pi/4} = \left(\frac{-1}{\sqrt{2}}\right) - \left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}-\sqrt{2}}{2}.$$

2/ Calculate the equation of tangent line to $y = \tan\theta$ at $\theta = \pi/4$.

$$\frac{dy}{d\theta} = \sec^2\theta = \frac{1}{\cos^2\theta} \Rightarrow \frac{dy}{d\theta}\Big|_{\theta=\frac{\pi}{4}} = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$y\big|_{\theta=\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) = 1 \Rightarrow$ tangent has slope 2
and contains point $(\pi/4, 1)$

$\Rightarrow \quad (y-1) = 2(\theta - \pi/4)$ is tangent equation.