Taylor Series (at 0)

<u>Aim</u> : Represent $f(x)$ as an <u>infinite series</u> for x near zero.

Let

$$\left.\begin{aligned} c_0 &= f(0) \\ c_1 &= \frac{f'(0)}{1!}x \\ c_2 &= \frac{f''(0)}{2!}x^2 \\ &\vdots \\ c_n &= \frac{f^{(n)}(0)}{n!}x^n \\ &\vdots \end{aligned}\right\}$$

n^{th} Taylor Polynomial of $f(x)$ at 0

$$p_n(x) = c_0 + c_1 + \dots + c_n$$

<u>Recall</u> : $f(x) \simeq p_n(x)$ with approximation improving as n increases (at least if x near 0)

$$\Rightarrow f(x) = c_0 + c_1 + c_2 + c_3 + \dots = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

equality

infinite series

<u>Taylor Series of $f(x)$ at 0</u>

So if we are lucky (ie f is a reasonable function) then $f(x)$ <u>equals</u> its Taylor series near 0.

Important Examples :

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ for all x (Amazing Fact)

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ for all x

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ for all x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ for x in $(-1,1)$ must be near 0

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$ for x in $(-1,1)$

geometric series.

From now on assume any function equals its Taylor Series near 0.

Can we determine a Taylor series without explicitly calculating it (very computationally intensive)?

Operations on Taylor Series

Let f and g have Taylor series

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$g(x) = b_0 + b_1 x + b_2 x^2 + \dots$

$\Rightarrow$ 1/ $f(x) + g(x)$ has Taylor series $(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots$

2/ $cf(x)$ has Taylor series $(ca_0) + (ca_1)x + (ca_2)x^2 + \dots$

constant

3/ $x^k f(x)$ ← the integer has Taylor Series $a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \dots$

4/ $f(x^k)$ has Taylor Series $a_0 + a_1 x^k + a_2 x^{2k} + \dots$

5/ $f'(x)$ has Taylor Series $a_1 + 2a_2 x + 3a_3 x^2 + \dots$ ← term by term derivative

6/ $\int f(x)dx$ has Taylor Series $C + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \dots$ ← term by term integral.

any constant

Examples

1/ What is the Taylor series of $h(x) = x^2 \sin(x^3)$? What is $h^{(5)}(0) = ?$

What is $h^{(20)}(0) = ?$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\Rightarrow \sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \dots$$

$$\Rightarrow x^2 \sin(x^3) = x^5 - \frac{x^{11}}{3!} + \frac{x^{17}}{5!} - \frac{x^{23}}{7!} + \dots$$ Taylor series of $h(x)$ at 0

$$\left(= h(0) + \frac{h'(0)}{1!}x + \frac{h''(0)}{2!}x^2 + \frac{h'''(0)}{3!}x^3 + \dots \right)$$

$$\Rightarrow 1 = \frac{h^{(5)}(0)}{5!}$$ ← The coefficients in front of x^5 must match

$$\Rightarrow h^{(5)}(0) = 5!$$

The coefficient in front of x^{20} is 0 $\Rightarrow$

$$0 = \frac{h^{(20)}(0)}{20!} \Rightarrow h^{(20)}(0) = 0$$

2/ What is the Taylor series of $\frac{1}{(1-x)^2}$? What is the Taylor series of $\ln((1-x)^{-1})$?

We could do these by brute force, repeatedly differentiating.

Instead observe that if $f(x) = \frac{1}{1-x} \Rightarrow f'(x) = \frac{1}{(1-x)^2}$ and

$$\int \frac{1}{1-x}\,dx = \frac{1}{-1}\ln(1-x) + C = \ln(1-x)^{-1} + C$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots$$ Term by term derivative

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

term by term integral

$$\Rightarrow \ln(1-x)^{-1} = \underbrace{\text{Some constant}}_{C} + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

Need to find C. Plug in 0 into both sides

$$\underbrace{\ln((1-0)^{-1})}_{=0} = C + 0 + 0 + 0 + \dots \Rightarrow C = 0$$

$$\Rightarrow \ln((1-x)^{-1}) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

<u>Nice Fact:</u> If $f(x)$ equals its Taylor series <u>for all</u> x $\Rightarrow$

same is true of $x^k f(x)$, $cf(x)$, $f(x^k)$, $f'(x)$ and $\int f(x)\,dx$.

<u>Example</u> Express $\int_0^1 e^{x^2}\,dx$ as an infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \underline{\text{for all } x}$$

$$\Rightarrow \quad e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots \quad \underline{\text{for all } x}$$

$$\Rightarrow \quad \int e^{x^2}\,dx = C + x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \dots \quad \underline{\text{for all } x}$$

$$\Rightarrow \quad \int_0^1 e^{x^2}\,dx = x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \dots \Big|_0^1$$

$$= 1 + \frac{1}{3} + \frac{1}{5 \cdot 2!} + \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} + \dots$$

<u>Example</u> What is the infinite series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

Note $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ <u>for all</u> x

$$\Rightarrow \quad e^{1} = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \dots$$

$$\Rightarrow \quad 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots = e$$

Pretty neat!