Taylor Polynomials

Aim : Approximate a function $f(x)$ with a polynomial near some specified number a.

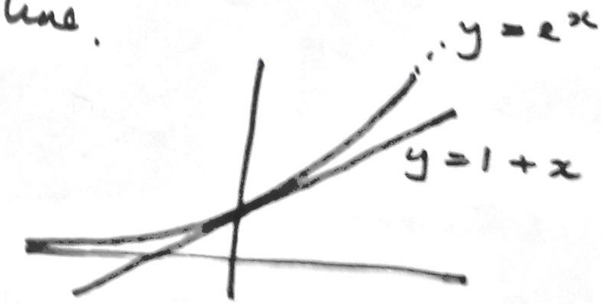
Example : $f(x) = e^x$, $a = 0$

Straight line ↙

Degree 1 : Need $p_1(x) = a_0 + a_1 x$ such that $y = p_1(x)$ approximats $y = e^x$ near 0. Choose tangent line.

$f(0) = e^0 = 1$

$f'(0) = e^0 = 1 \implies p_1(x) = 1 + x$



Problem : Only a good approximation very near 0. Must increase degree to improve approximation.

Degree 2 : Need $P_2(x) = a_0 + a_1 x + a_2 x^2$ such that $y = P_2(x)$ approximats $y = e^x$ near 0.

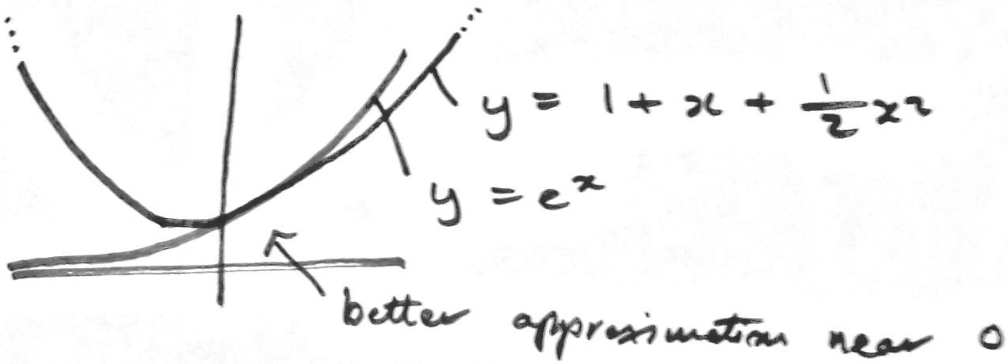
Obvious requirements : $\underset{\underset{a_0}{\|}}{P_2(0)} = f(0) = 1$ (same y-intercept)

$$\underset{a_1}{\underset{\parallel}{P_2'(0)}} = f'(0) = 1$$ (same slope at 0)

What about a_2? Let's also demand $f''(0) = P_2''(0)$.

$P_2''(0) = 2a_2 \quad f''(0) = 1 \implies a_2 = \frac{1}{2}$

$\implies P_2(x) = 1 + x + \frac{1}{2}x^2$



Degree 3 : Need $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $y = P_3(x)$ approximates $y = e^x$ near 0

Requirements :

$$\begin{aligned} a_0 &= P_3(0) = f(0) = 1 \\ a_1 &= P_3'(0) = f'(0) = 1 \\ 2a_2 &= P_3''(0) = f''(0) = 1 \\ 3\cdot 2\cdot a_3 &= P_3'''(0) = f'''(0) = 1 \end{aligned} \implies \begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_2 &= \frac{1}{2} \\ a_3 &= \frac{1}{3\cdot 2} \end{aligned}$$

$\Rightarrow P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2} \leftarrow$ Even better approximation near 0

Degree n : $p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

Requirements :

$$\left.\begin{aligned} a_0 &= p_n(0) = f(0) = 1 \\ a_1 &= p_n'(0) = f'(0) = 1 \\ 2 \cdot a_2 &= p_n''(0) = f''(0) = 1 \\ &\vdots \\ n \cdot (n-1) \ldots \cdot 2 \cdot a_n &= p_n^{(n)}(0) = f^{(n)}(0) = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_2 &= \frac{1}{2} \\ a_3 &= \frac{1}{3 \cdot 2} \\ &\vdots \\ a_n &= \frac{1}{n \cdot (n-1) \ldots 3 \cdot 2} \end{aligned}$$

Notation : $n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$

$\uparrow$ called "n factorial".

$\Rightarrow$ k^{th} coefficient is $\frac{1}{k!}$.

$\Rightarrow p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \leftarrow$ For big n this is a <u>very</u> good approximation near 0

Conclusions: To approximate $f(x)$ near 0 with

$P_n(x) = a_0 + a_1 x + \dots + a_n x^n$ we require

$$\begin{aligned} a_0 &= P_n(0) = f(0) \\ a_1 &= P_n'(0) = f'(0) \\ 2 \cdot a_2 &= P_n''(0) = f''(0) \\ 3!\, a_3 &= P_n'''(0) = f'''(0) \\ &\vdots \\ n!\, a_n &= P_n^{(n)}(0) = f^{(n)}(0) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a_0 &= f(0) \\ a_1 &= f'(0) = \frac{f'(0)}{1!} \\ a_2 &= \frac{f''(0)}{2} = \frac{f''(0)}{2!} \\ &\vdots \\ a_n &= \frac{f^{(n)}(0)}{n!} \end{aligned}$$

$$\Longrightarrow \quad P_n(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

$\nearrow$

Called n^{th} Taylor polynomial of $f(x)$ at 0.

General Principle: For reasonable $f(x)$, as long as we are near 0 the approximation gets better as n increases.

Fact: $f(x) \underset{\substack{\uparrow \\ \text{best } f}}{\approx} p_n(x)$

1/ x near 0

2/ n is big

Example $f(x) = e^x$, $P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{x^2}{2}$

$e^{0.1} = 1.105170918$, $e^{0.01} = 1.010050167$

$P_1(0.1) = 1.1$

$P_1(0.01) = 1.01$

$P_2(0.1) =$ [illegible]

1.105

$P_2(0.01) = 1.01005$

Example $f(x) = \sin(x)$ $\quad P_9(x) = ?$

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$ $\quad \Rightarrow$

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$\vdots$ repeats

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$ $\quad \Rightarrow$

$f'''(0) = -1$

$\vdots$ repeats

$a_0 = 0$

$a_1 = \frac{1}{1!}$

$a_2 = \frac{0}{2!}$

$a_3 = \frac{-1}{3!}$

$a_4 = \frac{0}{4!}$

$a_5 = \frac{1}{5!}$

$a_6 = \frac{0}{6!}$

$a_7 = \frac{-1}{7!}$

$a_8 = \frac{0}{8!}$

$a_9 = \frac{1}{9!}$

$$\Rightarrow P_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

What about approximating near some non-zero a?

Need $P_n(x)$, a degree n polynomial, with

$P_n(a) = f(a)$

$P_n'(a) = f'(a)$ $\quad \leftarrow$ just replacing 0 with a

$P_n''(a) = f''(a)$

$\vdots$

$P_n^{(n)}(a) = f^{(n)}(a)$

<u>Problem</u> : If we write $p_n(x) = a_0 + a_1 x + \dots + a_n x^n$

these conditions are <u>very</u> complicated if $a \neq 0$

<u>Clever trick</u> : Write $p_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$

(E.g. $1 + x + x^2 = 1 + ((x-1)+1) + ((x-1)+1)^2$
$(a=1)$

$= 3 + 3(x-1) + (x-1)^2$)

Now we have

$$a_0 = p_n(a) \qquad a_0 = f(a)$$

$$a_1 = p_n'(a) \qquad a_1 = f'(a) = \frac{f'(a)}{1!}$$

$$2 \cdot a_2 = p_n''(a) \quad \Longrightarrow \quad a_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$$

$$\vdots \qquad \vdots$$

$$n! \, a_n = p_n^{(n)}(a) \qquad a_n = \frac{f^{(n)}(a)}{n!}$$

$$\Rightarrow \quad P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\nearrow$

n^{th} Taylor polynomial of $f(x)$ at a.

__Example__ $f(x) = \sqrt{x}$, $a = 1$, $P_2(x) = ?$

$f(x) = \sqrt{x}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2} \cdot x^{-\frac{1}{2}} \Rightarrow f'(1) = \frac{1}{2}$

$f''(x) = \frac{-1}{4} \cdot x^{-\frac{3}{2}} \qquad f''(1) = \frac{-1}{4}$

$$\Rightarrow \quad P_2(x) = 1 + \frac{1/2}{1!}(x-1) + \frac{-1/4}{2!}(x-1)^2$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

So for x near 1 $\quad P_2(x) \approx \sqrt{x}$. E.g.

$$\sqrt{1.02} \simeq P_2(1.02) = 1 + \frac{1}{2} \cdot (0.02) - \frac{1}{8}(0.02)^2 = 1.00945$$

Example Using the 2^{nd} Taylor polynomial estimate the present value of a company with continuous income stream $e^{t(t+\frac{1}{2})}$ over $[0,1]$, where the annual interest rate is 50%.

$$\text{Present Value over } [0,1] = \int_0^1 e^{t(t+\frac{1}{2})} \cdot e^{-\frac{1}{2}t} \, dt = \int_0^1 e^{t^2} dt$$

Lets work out the 2^{nd} Talyor polynomial of e^{t^2} at 0.

$$e^{0^2} = 1$$

$$\frac{de^{t^2}}{dt} = 2te^{t^2} \Rightarrow \left.\frac{de^{t^2}}{dt}\right|_{t=0} = 0$$

$$\frac{d^2e^{t^2}}{dt^2} = \frac{d\,2te^{t^2}}{dt} = 2e^{t^2} + 4t^2e^{t^2} \Rightarrow \left.\frac{d^2e^{t^2}}{dt^2}\right|_{t=0} = 2$$

$$\Rightarrow P_2(t) = 1 + \frac{0}{1!}t + \frac{2}{2!}t^2 = 1+t^2$$

$$\Rightarrow e^{t^2} \approx 1+t^2 \quad (\text{near } 0)$$

$$\Rightarrow \int_0^1 e^{t^2} dt \approx \int_0^1 1+t^2 \, dt = t + \frac{1}{3}t^3 \Big|_0^1 = \frac{4}{3} \approx \begin{array}{c}\text{Present} \\ \text{Value over} \\ [0,1]\end{array}$$