Separable Differential Equations

A separable first order differential equation is one that can be written in the form $\frac{dy}{dx} = p(x)\, q(y)$. For example

$\frac{dy}{dx} = xy$, $\frac{dy}{dx} = \frac{x}{y} = x \cdot \left(\frac{1}{y}\right)$, $e^{x}\frac{dy}{dx} = y^2 \Rightarrow \frac{dy}{dx} = y^2 e^{-x}$.

<u>Aim</u> : Find general solution to separable equation.

1/ <u>Constant Solutions</u> : $y = k$ a constant solution $\Leftrightarrow$

$$\frac{dy}{dx} = 0 = p(x)\, q(y) = p(x)\, q(k) \Leftrightarrow q(k) = 0$$

ie <u>$y = k$ a solution if and only if $q(k) = 0$</u>

E.g. $\frac{dy}{dx} = x(y-1)$. $p(x) = x$, $q(y) = y - 1$

$q(k) = k - 1 = 0 \Leftrightarrow k = 1 \Rightarrow y = 1$ is <u>only</u> constant solution.

2/ Non-constant solutions : $\frac{dy}{dx} = p(x)\,q(y) \Rightarrow \frac{1}{q(y)}\,dy = p(x)\,dx$

$$\Rightarrow \int \frac{1}{q(y)}\,dy = \int p(x)\,dx$$

non-zero as not constant solution

Calculate both integrals and solve for y.

E.g. $\frac{dy}{dx} = x(y-1) \Rightarrow \frac{1}{y-1}\,dy = x\,dx \Rightarrow \int \frac{1}{y-1}\,dy = \int x\,dx$

$$\Rightarrow \ln|y-1| + C_1 = \frac{1}{2}x^2 + C_2$$ (C_1 and C_2 any constant)

$$\Rightarrow \ln|y-1| = \frac{1}{2}x^2 + C$$ ($C = C_2 - C_1$ any constant)

$$\Rightarrow |y-1| = e^{(\frac{1}{2}x^2 + C)} = e^C \cdot e^{\frac{1}{2}x^2}$$

$$\Rightarrow y - 1 = \pm e^C \cdot e^{\frac{1}{2}x^2}$$

$$\Rightarrow y = 1 \pm e^C \cdot e^{\frac{1}{2}x^2}$$

Let $A = \pm e^c$ then A can be any non-zero number.

$\Rightarrow$ Most general non-constant solution is

$$y = 1 + Ae^{\frac{1}{2}x^2} \quad (A \neq 0)$$

If we allow $A = 0$ we recover the only constant solution $y = 1$

$\Rightarrow$ $y = 1 + Ae^{\frac{1}{2}x^2}$ (A any constant) is most general solution.

<u>Example</u> Assume we have a bank account with fixed annual interest rate r. Let $y(t)$ = amount of money in account at time t. By definition $y' = ry$ ← Separable $p(x) = r$, $q(y) = y$

1/ $q(y) = 0 \Leftrightarrow y = 0 \Rightarrow y = 0$ is <u>only</u> constant solution

2/ $\frac{dy}{dt} = ry \Rightarrow \frac{1}{y}\,dy = r\,dt = \int \frac{1}{y}\,dy = \int r\,dt$

$\Rightarrow \ln|y| = rt + C \Rightarrow |y| = e^c \cdot e^{rt}$

$\Rightarrow \quad y = \pm e^c \cdot e^{rt}$ (c any constant)

$\Rightarrow \quad y = A e^{rt}$ (A any non-zero constant)

most general non-constant solution.

$\Rightarrow \quad y = A \cdot e^{rt}$ (A any constant) most general solution.

$y(0) = A \Rightarrow A =$ amount in account at $t = 0$.

This rigorously proves what you were told in 16A. Neat!

There's no real need to simplify in this way. It just makes the final answer easier to deal with.

<u>Example</u> Find the solution to $\frac{dy}{dx} = \frac{x}{y}$ with initial condition $y(1) = -6$.

$p(x) = x$, $q(y) = 1/y$ $\quad q(y) = 0$ has <u>no</u> solutions

$\Rightarrow$ No constant solutions

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx \Rightarrow$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C \Rightarrow y^2 = x^2 + 2C \Rightarrow y = \pm\sqrt{x^2+2C}$$

$\Rightarrow$ general solution is $y = \pm\sqrt{x^2+2C}$ (C arbitrary constant)

$$y(1) = \pm\sqrt{1+2C} = 6 \Rightarrow \sqrt{1+2C} = 6 \Rightarrow 2C = 35$$

choose +

$$\Rightarrow \quad y = +\sqrt{x^2+35}.$$

<u>Example</u> A company has found that the rate at which a person new to the assembly line produces items is

$$\frac{dy}{dx} = 7.5e^{-0.3y} \quad (y = \text{number of units produced per day})$$

where x is the number of days on the line. How many items can a worker be expected to produce on the 8th day, assuming $y(0) = 0$.

$p(x) = 7.5 \,,\quad q(y) = e^{-0.3y}$

$q(y) = 0$ has no solutions $\Rightarrow$ No constant solutions

$$\frac{dy}{dx} = 7.5\, e^{-0.3y} \Rightarrow \int e^{0.3y}\, dy = \int 7.5\, dx$$

$$\Rightarrow \frac{1}{0.3} e^{0.3y} = 7.5x + C$$

$$y(0) = 0 \Rightarrow \frac{1}{0.3} e^{0} = (7.5)\cdot 0 + C$$

$$\Rightarrow \frac{1}{0.3} = C$$

$$\Rightarrow \frac{1}{0.3} e^{0.3y} = 7.5x + \frac{1}{0.3}$$

$$\Rightarrow e^{0.3y} = (2.25)x + 1$$

$$\Rightarrow y = \frac{1}{0.3} \ln((2.25)x + 1)$$

$$y(8) = \frac{1}{0.3} \ln((2.25)\cdot 8 + 1) \approx 10$$

Example Solve $y' - \frac{y}{x} = \frac{-1}{x}$ with $x > 0$ and

a) $y(4) = 8$ and b) $y(-1) = 1$

$y' - \frac{y}{x} = \frac{-1}{x} \Rightarrow \frac{dy}{dx} = \frac{y-1}{x}$. Let $p(x) = 1/x$
$q(y) = y - 1$

$q(y) = 0 \Leftrightarrow y - 1 = 0 \Leftrightarrow y = 1 \Rightarrow y = 1$ is only constant sol.

$\frac{dy}{dx} = \frac{y-1}{x} \Rightarrow \int \frac{1}{y-1}\,dy = \int \frac{1}{x}\,dx \Rightarrow \ln|y-1| = \ln|x| + C$

$\Rightarrow y - 1 = \pm e^C \cdot |x| \Rightarrow y = 1 \pm e^C \cdot |x| \overset{(x>0)}{=} 1 \pm e^C \cdot x$

We can simplify this to $y = 1 + Ax$

If we let A be any constant this includes both constant and non-constant solutions.

a) $y(4) = 8 \Rightarrow 1 + 4A = 8 \Rightarrow A = \frac{7}{4} \Rightarrow y = 1 + \frac{7}{4}x$

b) $y(-1) = 1 \Rightarrow 1 - A = 1 \Rightarrow A = 0 \Rightarrow y = 1$.

Remark : If we hadn't simplified our answer it would have been much more confusing to find initial condition solutions. You don't have to do it but it certainly helps.