Review

Integration by Substitution : $\int f(g(x))\,g'(x)\,dx = ?$

Let $u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)} \Rightarrow$

$$\int f(g(x))\,g'(x)\,dx = \int f(g(x))\,g'(x)\,\frac{du}{g'(x)} = \int f(g(x))\,du = \int f(u)\,du$$

$$= F(u) + C = F(g(x)) + C$$

antiderivative of f

Important Examples :

$\int f(x^n) \cdot x^{n-1}\,dx \Rightarrow u = x^n$

$\int f(\sin(x))\cos(x)\,dx \Rightarrow u = \sin(x)$

$\int f(\ln(x)) \cdot 1/x\,dx \Rightarrow u = \ln(x)$

$\int \frac{g'(x)}{g(x)}\,dx \Rightarrow u = g(x)$

$\int e^{g(x)} g'(x)\,dx \Rightarrow u = g(x)$

Useful Fact :

$\int f(x)\,dx = F(x) + C$

$\Rightarrow \int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C$

(shown using $u = ax+b$)

Example $\int \sqrt{x}\,\sin(x^{3/2})\,dx = ?$ Let $u = x^{3/2}$

$\Rightarrow \frac{du}{dx} = 3/2\, x^{1/2} = \frac{3}{2}\sqrt{x} \Rightarrow dx = \frac{2}{3} \cdot \frac{du}{\sqrt{x}}$

$$\Rightarrow \int \sqrt{x}\,\sin(x^{3/2})\,dx = \int \frac{2}{3}\sin(u)\,du = \frac{-2}{3}\cos(u) + C$$

$$= \frac{-2}{3}\cos(x^{3/2}) + C$$

<u>Hard Example</u> : $\int \frac{2x\cos(\ln(x^2+1))}{x^2+1}\,dx = ?$

$$u = \ln(x^2+1) \Rightarrow \frac{du}{dx} = \frac{2x}{x^2+1} \Rightarrow dx = \frac{(x^2+1)}{2x}\,du$$

$$\Rightarrow \int \frac{2x\cos(\ln(x^2+1))}{x^2+1}\,dx = \int \cos(u)\,du = \sin(u) + C$$

$$= \sin(\ln(x^2+1)) + C$$

<u>Integration by Parts</u> : $\int f(x)\,g(x)\,dx = ?$

$$G'(x) = g(x) \Rightarrow \int f(x)\,g(x)\,dx = f(x)\,G(x) - \int f'(x)\,G(x)\,dx$$

Important Examples :

$$\int x^n e^x\,dx \Rightarrow f(x) = x^n,\; g(x) = e^x$$

$$\int x^n \ln(x) \Rightarrow f(x) = \ln(x),\; g(x) = x^n$$

$\int x^n \sin(x)/\cos(x)\, dx \Rightarrow f(x) = x^n \quad g(x) = \sin(x)/\cos(x)$

<u>Example</u> $\int \ln(x)\, dx = ? \quad f(x) = \ln(x),\ g(x) = 1$
$f'(x) = \frac{1}{x},\ G(x) = x \Rightarrow$

$\int \ln(x)\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx = x \ln(x) - x + C$

<u>Hard Example</u> $\int (\ln(x))^2 dx = ? \quad f(x) = (\ln(x))^2,\ g(x) = 1$
$f'(x) = 2\frac{\ln(x)}{x},\ G(x) = x$

$\Rightarrow \int (\ln(x))^2 dx = x(\ln(x))^2 - 2\int \ln(x)\, dx$

$= x(\ln(x))^2 - 2x\ln(x) + 2x + C$

Exercise try u-sub : $u = \ln(x)$.

When doing <u>definite integrals</u>, first calculate indefinite integral.

<u>Example</u> : $\int_0^7 x e^{x^2} dx = ?$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow \int x e^{x^2} dx = \frac{1}{2}\int e^u du$

$= \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$

$$\Rightarrow \int_0^2 x e^{x^2}\, dx = \frac{1}{2} e^{x^2} \Big|_0^2 = \frac{1}{2} e^4 - \frac{1}{2}.$$

<u>Improper Integrals</u>: $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx, \quad \int_{-\infty}^\infty f(x)\,dx = \int_0^\infty f(x)\,dx + \int_{-\infty}^0 f(x)\,dx$$

Convergent $\Leftrightarrow$ all limits exist.

<u>Example</u> $\displaystyle\int_{-\infty}^0 \frac{x}{(x^2+1)^2}\,dx = ?$

Let $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$$\Rightarrow \int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = \frac{-1}{2u} + c$$

$$\Rightarrow \frac{-1}{2(x^2+1)} + c$$

$$\Rightarrow \int_t^0 \frac{x}{(x^2+1)^2}\,dx = \frac{-1}{2(x^2+1)}\Big|_t^0 = \frac{-1}{2} + \frac{1}{2(t^2+1)}$$

$$\Rightarrow \int_{-\infty}^0 \frac{x}{(x^2+1)^2}\,dx = \lim_{t \to -\infty}\left(\frac{-1}{2} + \frac{1}{2(t^2+1)}\right) = \frac{-1}{2}$$

Integration by Parts : $\int f(x)\,g(x)\,dx = ?$

$$G'(x) = g(x) \Rightarrow \int f(x)\,g(x)\,dx = f(x)\,G(x) - \int f'(x)\,G(x)\,dx$$

Core Examples

1/ $\int x^n e^x \, dx$ (n positive whole number) $\Rightarrow f(x) = x^n, \; g(x) = e^x$

2/ $\int x^n \sin(x) \, dx$ ($\cos(x)$) $\Rightarrow f(x) = x^n, \; g(x) = \sin(x)$ ($\cos(x)$)

3/ $\int x^a \ln(x)$ (a any number) $\Rightarrow f(x) = \ln(x) \quad g(x) = x^a$

Example

1/ $\int \ln(x)\,dx \Rightarrow$ $f(x) = \ln(x) \quad g(x) = 1$

$f'(x) = \frac{1}{x} \quad G(x) = x$

$$= \int \ln(x)\,dx = x\ln(x) - \int \frac{1}{x} \cdot x \, dx = x\ln(x) - x + c$$

Harder Example : $\int (\ln(x))^2 \, dx = ?$

$f(x) = (\ln(x))^2 \qquad g(x) = 1$

$f'(x) = \dfrac{2\ln(x)}{x} \qquad G(x) = x$

$\Rightarrow \int (\ln(x))^2 \, dx = x(\ln(x))^2 - 2\int \ln(x) \, dx$

$= x(\ln(x))^2 - 2x\ln(x) + 2x + C$

Important Applications :

Accumulated Value over $[0,T]$ $= \int_0^T \underbrace{f(t)}_{\text{income rate}} e^{\overbrace{r}^{\text{interest rate}}(T-t)} \, dt$ (Bank account starts with \$0)

Present Value of income stream over $[0,T]$ $= \int_0^T f(t) \, e^{-rt} \, dt$

Capital Value $= \int_0^\infty f(t) \, e^{-rt} \, dt$

Population within radius r of center $= \int_0^r \underbrace{D(t)}_{\text{pop. density at distance } t \text{ from center.}} 2\pi t \, dt$

<u>Example</u> $f(t) = \dfrac{2}{1+e^{-\frac{1}{2}t}}$ $r = 0.5$ $\Rightarrow$ Capital Value = ?

$$\int f(t)\, e^{-rt}\, dt = \int \frac{2e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\, dt$$

$$u = 1 + e^{-\frac{1}{2}t} \Rightarrow \frac{du}{dt} = -\frac{1}{2}e^{-\frac{1}{2}t} \Rightarrow dt = \frac{-2du}{e^{-\frac{1}{2}t}}$$

$$\Rightarrow \int \frac{2e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}} = -4\int \frac{1}{u}\, du = -4\ln|u| + c$$

$$= -4\ln|1+e^{-\frac{1}{2}t}| + c$$

$$\Rightarrow \int_0^b f(t)\, e^{-rt}\, dt = -4\ln|1+e^{-\frac{1}{2}t}| \Big|_0^b = 4\ln(2) - 4\ln(1+e^{-\frac{1}{2}b})$$

$$\int_0^\infty \frac{2e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\, dt = \lim_{b\to\infty} \left(4\ln(2) - 4\ln(1+e^{-\frac{1}{2}b})\right)$$

$$= 4\ln(2) - 4\ln(1+0) = 4\ln(2)$$

$$\Rightarrow C.V. = 4\ln(2)$$

Differential Equations Review

$$\frac{dy}{dx} = p(x)\, q(y) \quad - \quad \text{Separable equation.}$$

Aim: Find general solution.

1/ $q(y) = 0 \Rightarrow$ solutions give only <u>constant</u> solutions

2/ $\int \frac{1}{q(y)}\, dy = \int p(x)\, dx \Rightarrow$ Gives all <u>non-constant</u> solutions.

$$\frac{dy}{dx} + a(x)\, y = b(x) \quad - \quad \text{Linear equation}$$

Aim: Find general solution. remember + C

$$y(x) = \frac{1}{e^{A(x)}} \left(\int e^{A(x)}\, b(x)\, dx \right)$$ gives <u>all</u> solutions.

$$\frac{dy}{dx} = q(y) \quad - \quad \text{Autonomous equation.}$$

Aim: If cannot calculate $\int \frac{1}{q(y)}\, dy$ graph solutions

1/ Plot $z = q(y)$

2/ On y-axis (in xy-plane) plot lines with slope $z = g(y)$.

3/ Draw solutions following these guide lines.

Important Applications :

$$y(t) = \begin{cases} \text{account balance at time } t \\ \text{population at time } t \\ \text{loan balance at time } t \end{cases}$$

$$\Rightarrow \frac{dy}{dt} = ky + f(t) - g(t)$$

k: $\begin{cases} \text{interest rate} \\ \text{growth constant} \\ \text{interest rate} \end{cases}$ $f(t)$: $\begin{cases} \text{income rate} \\ \text{immigration rate} \\ \text{\_\_\_\_} \end{cases}$ $g(t)$: $\begin{cases} \text{withdrawal rate} \\ \text{emigration rate} \\ \text{payment rate} \end{cases}$

Examples

1/ $\dfrac{dy}{dt} = \dfrac{1}{t\ln(t)}(1-y)$ Find general solution. (assume $t > 1$)

Both separable and linear.

<u>Separable Method</u>: $p(t) = \frac{1}{t\ln(t)}$, $q(y) = 1-y$

Constant Solutions : $q(y) = 0 \Rightarrow 1-y = 0 \Rightarrow y = 1$

$\Rightarrow y(t) = 1$ is only constant solution.

Non constant Solutions : $\int \frac{1}{1-y}\,dy = \int \frac{1}{t\ln(t)}\,dt$

$u = \ln(t) \Rightarrow \frac{du}{dt} = \frac{1}{t} \Rightarrow dt = t\,du$

$\Rightarrow \int \frac{1}{t\ln(t)}\,dt = \int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|\ln(t)| + C \underset{t>1}{\overset{\uparrow}{=}} \ln(\ln(t)) + C$

$\Rightarrow -\ln|1-y| = \ln(\ln(t)) + C$

$\Rightarrow \ln(|1-y|^{-1}) = \ln(\ln(t)) + C$

$\Rightarrow |1-y|^{-1} = \ln(t) \cdot e^{C}$

$\Rightarrow |1-y| = \frac{e^{-C}}{\ln(t)} \Rightarrow 1-y = \frac{\pm e^{-C}}{\ln(t)}$

$$\Rightarrow \quad y = 1 - \frac{\pm e^{-c}}{\ln(t)} \quad \left(\begin{array}{l} c \text{ any constant} \\ \text{Either } + \text{ or } - \end{array} \right)$$

$$\Rightarrow \quad y = \begin{cases} 1 \\ 1 - \dfrac{\pm e^{-c}}{\ln(t)} \end{cases} \quad \text{a general solution.}$$

<u>Linear Method</u> : $\dfrac{dy}{dt} = \dfrac{1}{t\ln(t)}(1-y)$

$$\Rightarrow \quad \frac{dy}{dt} + \frac{1}{t\ln(t)}\, y = \frac{1}{t\ln(t)} \quad \Rightarrow \quad \begin{array}{l} a(t) = \dfrac{1}{t\ln(t)} \\ b(t) = \dfrac{1}{t\ln(t)} \end{array}$$

$$\Rightarrow \quad A(t) = \ln|\ln(t)| \underset{t>1}{=} \ln(\ln(t)) \quad \Rightarrow e^{A(t)} = \ln(t)$$

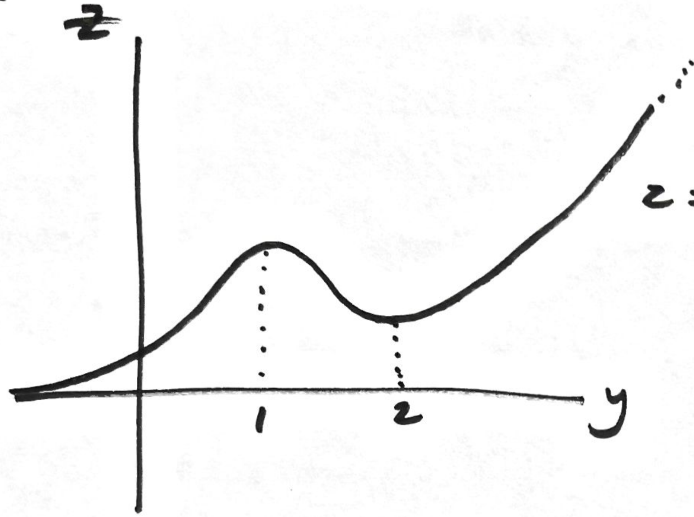
just done

$$\Rightarrow \quad y(t) = \frac{1}{\ln(t)} \cdot \int \ln(t) \cdot \frac{1}{t\ln(t)}\, dt = \frac{1}{\ln(t)} \left(\int \frac{1}{t}\, dt \right)$$

$$= \frac{1}{\ln(t)}\left(\ln(t) + c\right) = 1 + \frac{c}{\ln(t)} \quad (c \text{ arbitrary})$$

$$\Rightarrow \quad y(t) = 1 + \frac{c}{\ln(t)} \quad \text{general solution.}$$

2/ (Graphic Methods) $\frac{dy}{dx} = q(y)$



$z = q(y)$ $\rightsquigarrow$