Review

Laws of Derivatives : $\frac{d}{dx}(x^r) = r x^{r-1}$, $\frac{d}{dx} e^x = e^x$,

$\frac{d}{dx} \ln(x) = \frac{1}{x}$, $\frac{d}{dx}(f(x) + g(x)) = \frac{d f(x)}{dx} + \frac{d g(x)}{dx}$.

$\frac{d}{dx}(\overset{\text{constant}}{k} f(x)) = k \frac{d}{dx} f(x)$. $\frac{d f(x) g(x)}{dx} = f'(x) g(x) + f(x) g'(x)$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) g(x) - f(x) g'(x)}{(g(x))^2}$, $\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$

$\frac{d}{dx} \sin(x) = \cos(x)$, $\frac{d}{dx} \cos(x) = -\sin(x)$, $\frac{d}{dx} \tan(x) = \sec^2(x)$

You'll need to know all these to calculate partial derivatives.

Example

$f(x, y, z) = \sin(xy) + e^{zy} \Rightarrow$

$\frac{\partial f}{\partial x} = y \cos(xy)$, $\frac{\partial f}{\partial y} = x \cos(xy) + z e^{zy}$, $\frac{\partial f}{\partial z} = y e^{zy}$.

Make sure you can evaluate $\sin(x)$, $\cos(x)$, $\tan(x)$ at $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}$, etc.

Unconstrained Optimization : Find rel. max/min of $f(x,y)$.

1^{st} Derivative Test : Solve $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$. This gives critical points, ie potential max/mins

2^{nd} Derivative Test : Calculate $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and

$$D(x,y) = \frac{\partial^2 f}{\partial x} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

Evaluate $D(a,b)$, where (a,b) critical.

$D(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0 \Rightarrow$ min

$\frac{\partial^2 f}{\partial x^2}(a,b) < 0 \Rightarrow$ max

$D(a,b) < 0 \Rightarrow$ Saddle, $D(a,b) = 0 \Rightarrow$ inconclusive

Example

$f(x,y) = x^3 - y^3 - 3xy$

1/ $\frac{\partial f}{\partial x} = 3x^2 - 3y = 0$

$\frac{\partial f}{\partial y} = -3y^2 - 3x = 0$

Solve one in a single variable

$\Rightarrow y = x^2 \Rightarrow -3x^4 - 3x = 0$

Sub into other

$\Rightarrow \quad -3x(x^3+1) = 0 \Rightarrow x = 0$ or $x = -1$

$\Rightarrow \quad y = 0$ or $1 \Rightarrow (0,0), (-1,1)$ only critical points

2/ $\dfrac{\partial^2 f}{\partial x^2} = 6x, \quad \dfrac{\partial^2 f}{\partial y^2} = -6y, \quad \dfrac{\partial^2 f}{\partial x \partial y} = -3 \Rightarrow$

$D(x,y) = -36xy - 9$

$D(0,0) = -9 \Rightarrow (0,0)$ saddle

$D(-1,1) = 36 - 4 = 25 > 0$ <u>and</u> $\dfrac{\partial^2 f}{\partial x^2}(-1,1) = -6 < 0 \Rightarrow$ max

<u>Constrained Optimization (Lagrange Multipliers)</u>:

Find max/min of $f(x,y)$ subject to $g(x,y) = 0$.

Method:

1/ Define $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

2/ Solve $\dfrac{\partial F}{\partial x} = 0, \ \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial \lambda} = 0 \leftarrow$ always $g(x,y)$.

3/ (a,b,c) solution $\Rightarrow$ (a,b) potential max / min

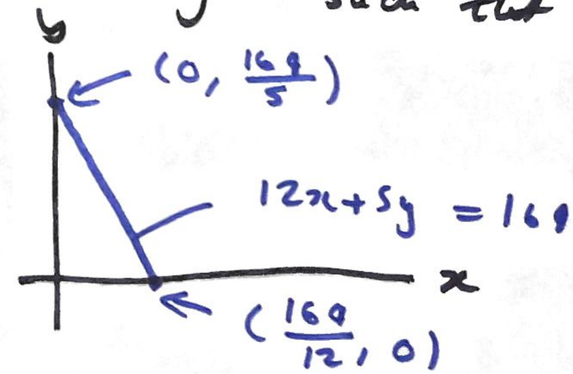
This gives <u>all</u> potential max/min locations.

In general <u>no</u> 2^{nd} derivative test.

<u>Special Case</u> : Constraint $g(x,y)=0$ has endpoints. Then find all above (a,b), calculate $f(a,b)$. Also calculate f at endpoints. This will find <u>absolute</u> max / min.

<u>Example</u> Minimize $f(x,y) = x^2+y^2$ such that

$12x + 5y = 169$, where $x, y \geq 0$.



$g(x,y) = 12x + 5y - 169$

$\Rightarrow F(x,y,\lambda) = x^2+y^2 + 12\lambda x + 5\lambda y - 169$

$\frac{\partial F}{\partial x} = 2x + 12\lambda = 0$

$\frac{\partial F}{\partial y} = 2y + 5\lambda = 0$

solve for λ

$\Rightarrow \lambda = \frac{-2x}{12} = \frac{-2y}{5} \Rightarrow y = \frac{5}{12}x$

$$\frac{\partial F}{\partial \lambda} = 12x + 5y - 169 = 0 \Rightarrow 12x + 5\frac{5}{12}x = 169$$

substitute

$$\Rightarrow \frac{12^2 + 5^2}{12} x = 169 \Rightarrow \frac{169}{12} x = 169 \Rightarrow x = 12$$

$$\Rightarrow y = 5 \Rightarrow \lambda = -2$$

$\Rightarrow$ $(12, 5)$ only critical / Lagrange point

$$f(12,5) = 169$$

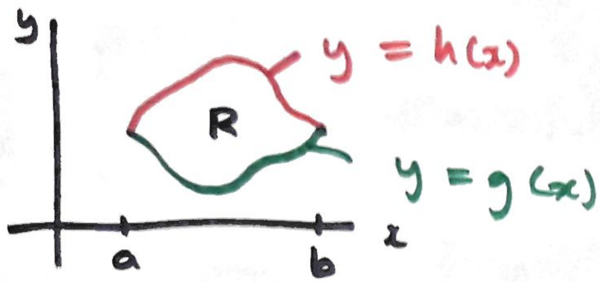
$$f\left(0, \frac{169}{5}\right) = \frac{(169)^2}{5^2} = 169 \cdot \frac{169}{25} > 169$$

$$f\left(\frac{169}{12}, 0\right) = \left(\frac{169}{12}\right)^2 = 169 \cdot \frac{169}{144} > 169$$

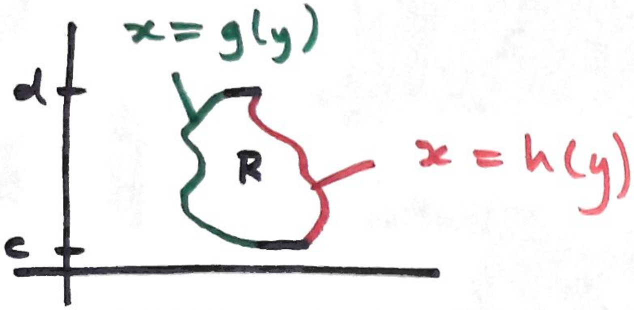
$\Rightarrow$ $\underset{169}{\overset{\shortparallel}{f(12,5)}}$ is <u>min value</u> of $x^2 + y^2$ when $12x + 5y - 169 = 0$, $x, y \geq 0$.

Double Integrals

Notation / Method of computation



$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

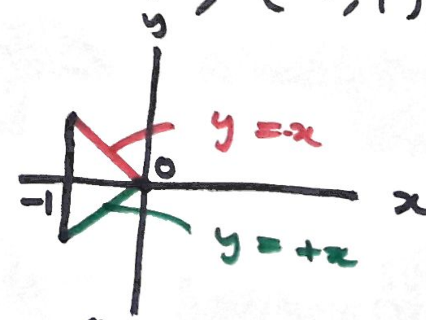


$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$

Can often take either approach. Choose the easiest.

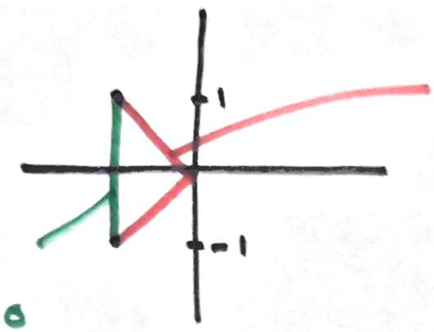
Example R = triangle with corners $(0,0)$, $(-1,1)$, $(-1,-1)$

Calculate $\displaystyle\iint_R x \; dx\,dy$.



$$\int_x^{-x} x\,dy = xy\Big|_x^{-x} = -2x^2 \Rightarrow \iint_R x\,dx\,dy = \int_{-1}^{0} -2x^2\,dx = \frac{-2}{3}x^3\Big|_{-1}^{0} = \frac{-2}{3}$$

Could have tried :



$$x = \begin{cases} -y & \text{if } y \geq 0 \\ y & \text{if } y < 0 \end{cases}$$

The method would still work but $h(y)$ is more complicated so integral is harder.