Poisson and Geometric Random Variables

X – D.R.V. with possible values $0, 1, 2, 3, \ldots$

For every $n \geq 0$ a whole number we have probabilities $\Pr(X = n)$.

Properties

1/ $\Pr(X = n) \geq 0$ for all $n \geq 0$

infinite series

2/ $\Pr(X=0) + \Pr(X=1) + \Pr(X=2) + \ldots = 1$

Recall:

$$\mathbb{E}(X) = 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) + 2 \cdot \Pr(X=2) + \ldots$$

$$\mathrm{Var}(X) = (0 - \mathbb{E}(X))^2 \Pr(X=0) + (1 - \mathbb{E}(X))^2 \cdot \Pr(X=1) + (2 - \mathbb{E}(X))^2 \Pr(X=2) + \ldots$$

Examples

1/ X = number of calls a telephone switchboard gets in a minute

2/ If we repeatedly toss a coin

X = number of successive tails before a head.

Poisson Random Variable

X, a D.R.V. with possible values $0, 1, 2, 3, \ldots$

is **Poisson** if $\Pr(X = n) = \dfrac{\lambda^n}{n!} e^{-\lambda}$ ($\lambda > 0$ a fixed constant)

(Convention : $0! = 1 \Rightarrow \Pr(X = 0) = e^{-\lambda}$)

1/ $\dfrac{\lambda^n}{n!} e^{-\lambda} > 0$ for all $n \geq 0$ as $\lambda > 0$

2/ $\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \ldots$

$$= e^{-\lambda} + \frac{\lambda}{1!} e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda} + \frac{\lambda^3}{3!} e^{-\lambda} + \ldots$$

$$= e^{-\lambda} \underbrace{\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \right)}_{e^{\lambda} \text{ (Taylor series)}}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1$$

<u>Facts</u> $\mathbb{E}(X) = \lambda$, $Var(X) = \lambda$

<u>Example</u> X = number of calls to switchboard in a minute

Assume X Poisson. If the average number of calls is 5, what is the probability that 3 or more calls a taken in a minute?

$$Pr(X = n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{(Poisson)}$$

$$\mathbb{E}(X) = 5 \Rightarrow \lambda = 5$$

$$\begin{aligned} Pr(X \geq 3) &= 1 - Pr(X < 3) \\ &= 1 - Pr(X=2) - Pr(X=2) - Pr(X=0) \\ &= 1 - e^{-5} - \frac{5}{1!} e^{-5} - \frac{5^2}{2!} e^{-5} \approx 0.8755 \end{aligned}$$

<u>Geometric Random Variables</u>

X a D.R.V. with possible values $0, 1, 2, 3, \ldots$

is <u>geometric</u> if $Pr(X=n) = (1-p)p^n$ for $n \geq 0$ $\left(\begin{array}{l} p < 1 \\ p \geq 0 \end{array}\right)$

1/ $0 \le p < 1 \Rightarrow (1-p)p^n \geq 0$ for all $n \geq 0$

2/ $Pr(X=0) + Pr(X=1) + Pr(X=2) + \dots$

$$= (1-p) + (1-p)p + (1-p)p^2 + (1-p)p^3 + \dots$$

$$= (1-p)\big(\underbrace{1 + p + p^2 + p^3 + \dots}_{\frac{1}{1-p} \text{ (geometric series)}}\big)$$

$$= \frac{(1-p)}{(1-p)} = 1$$

<u>Facts</u> : $\mathbb{E}(X) = \dfrac{p}{1-p}$, $Var(X) = \dfrac{p}{(1-p)^2}$

<u>Example</u> 1/ X a geometric random variable with expected value 2. Calculate $Pr(X \le 4)$.

Geometric $\Rightarrow$ $Pr(X=n) = (1-p)p^n$ $(0 \le p < 1)$

$$E(X) = \frac{p}{1-p} = 2 \Rightarrow p = 2-2p \Rightarrow 3p = 2$$

$$\Rightarrow p = \frac{2}{3} \Rightarrow 1-p = \frac{1}{3}$$

$$Pr(X \leq 4) = Pr(X=0) + Pr(X=1) + Pr(X=2) + Pr(X=3) + Pr(X=4)$$

$$= \frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3}\left(\frac{2}{3}\right)^4$$

(Recall : $a + ar + ar^2 + \dots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$) $\quad a = \frac{1}{3}$, $r = \frac{2}{3}$

$$= \frac{\frac{1}{3}\left(1-\left(\frac{2}{3}\right)^5\right)}{1-\frac{2}{3}} = \left(1-\left(\frac{2}{3}\right)^5\right)$$

<u>Example</u> A group of space marines are under attack by aliens. After the initial attack they are attacked at hourly intervals. A marine has an 80% chance of surviving an attack. How long can a marine expect to live?

X = survival time for a marine 20% $\rightarrow$

$X = 0 \Rightarrow$ marine dies in 1st attack $\Rightarrow Pr(X=0) = \frac{1}{5}$

$X = 1 \Rightarrow$ marine survives 1st attack and dies in 2nd attack $\Rightarrow Pr(X=1) = \frac{4}{5} \cdot \frac{1}{5}$

survive 1st Die in 2nd

$X = 2 \Rightarrow$ Marine survives 1st 2 attacks and dies in 3rd.

$\Rightarrow P(X=2) = \left(\frac{4}{5}\right)^2 \cdot \left(\frac{1}{5}\right)$

$X = n \Rightarrow$ Marine survives 1st n attacks and dies in n+1 attack

$\Rightarrow P(X=n) = \left(\frac{4}{5}\right)^n \left(\frac{1}{5}\right)$

$\Rightarrow X$ geometric with $p = \frac{4}{5} \Rightarrow \mathbb{E}(X) = \dfrac{\frac{4}{5}}{1-\frac{4}{5}} = 4$

A marine can expect to survive 4 hours.