Partial Derivatives

"rate of change of f with respect to x" $= \dfrac{dy}{dx}$

$y = f(x)$

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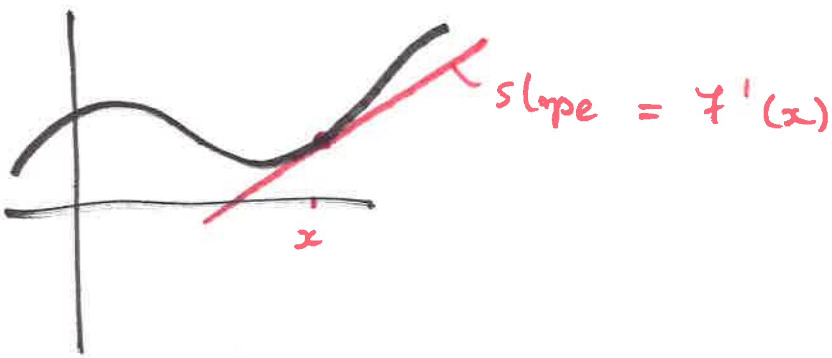
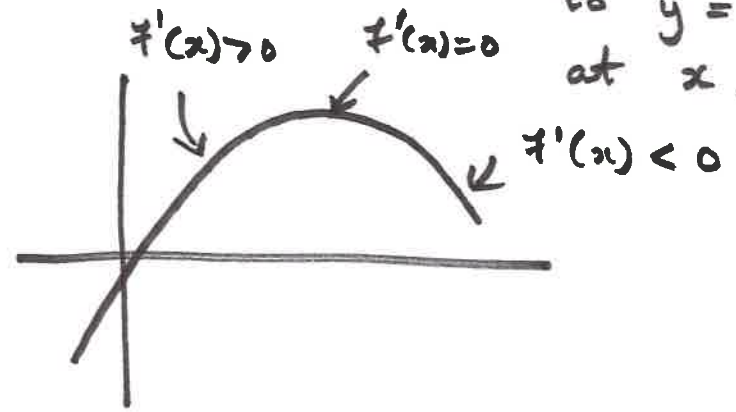
Derivative of f with respect to x $= f'(x) = \dfrac{df}{dx} =$ Slope of tangent line to $y = f(x)$ at x.



$\Rightarrow$



Aim: Generalize to multivariable functions.

Example A small company makes 2 products, smartphones and tablets.

x = number of smartphones sold

y = number of tablets sold

Profit $= P(x,y) = 40x^2 - 10xy + 5y^2 - 80$

How will a change in x or y affect P?

Suppose sales of smartphones are steady at 10 units ($\Rightarrow x = 10$) and only y varies $\Rightarrow$

$$\text{Profit} = f(y) = P(10, y) = 40 \cdot 10^2 - 10 \cdot 10y + 5y^2 - 80$$
$$= 3920 - 100y + 5y^2$$

$\Rightarrow$ Rate of change of P with respect to y, fixing $x = 10$ $= \dfrac{df}{dy} = -100 + 10y$

Conversely, suppose sales of tablets are steady at 10 units ($\Rightarrow y = 10$) and only x varies $\Rightarrow$

$$\text{Profit} = g(x) = P(x, 10) = 40x^2 - 10 \cdot x \cdot 10 - 5 \cdot 10^2 - 80$$
$$= 40x^2 - 100x - 580$$

$\Rightarrow$ Rate of change of P with respect to x, fixing $y = 0$ $= \dfrac{dg}{dx} = 80x - 100$

Q/ If the company has fixed sales $x = 10$, $y = 5$, will increasing y (while keeping $x = 10$) a small amount increase or decrease profit?

$$\frac{df}{dy} = -100 + 10y \Rightarrow \left.\frac{df}{dy}\right|_{y=5} = -100 + 50 = -50 < 0$$

"evaluated at $y = 5$"

$\Rightarrow$ Increasing y will decrease profit.

<u>Definition</u> Let $f(x, y)$ be a 2-variable function.

$\frac{\partial f}{\partial x}$ = <u>Partial</u> Derivative of f with respect to x = Derivative of f obtained by treating x as a variable and y as a constant

$\frac{\partial f}{\partial y}$ = <u>Partial</u> Derivative of f with respect to y = Derivative of f obtained by treating y as a variable and x as a constant.

<u>Remark</u> $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are <u>2-variable functions</u>.

In the first case we don't need to replace y with a specific number (e.g. $y = 10$). We just leave it as y and pretend its a constant.

Examples

1/ $P(x, y) = 40x^2 - 10xy + 5y^2 - 80 \Rightarrow$

$$\frac{\partial P}{\partial x} = 40 \cdot 2 \cdot x - 10y = 80x - 10y$$

$$\frac{\partial P}{\partial y} = -10x + 10y$$

Note that $\left.\frac{\partial P}{\partial x}\right|_{y=10} = 80x - 100$ and $\left.\frac{\partial P}{\partial y}\right|_{x=10} = -100 + 10y$

as expected.

2/ $f(x, y) = e^{xy^2} \Rightarrow \frac{\partial f}{\partial x} = ?$, $\frac{\partial f}{\partial y} = ?$

y constant $\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial xy^2}{\partial x} e^{xy^2} = y^2 e^{xy^2}$

Chain

x constant $\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial\, xy^2}{\partial y}\, e^{xy^2} = 2xy\, e^{xy^2}$

3/ Evaluate $\dfrac{\partial f}{\partial x}$ at $(1,2)$ where $f(x,y) = \ln\left(\dfrac{x+y}{y}\right)$

Chain Rule : $\dfrac{\partial f}{\partial x} = \dfrac{\left(\dfrac{\partial\left(\frac{x+y}{y}\right)}{\partial x}\right)}{\left(\frac{x+y}{y}\right)} = \dfrac{\left(\frac{1}{y}\right)}{\left(\frac{x+y}{y}\right)} = \dfrac{1}{x+y}$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{\substack{x=1\\ y=2}} = \frac{1}{3}.$$

The concept of partial derivative also makes sense of more variable functions. E.g. If f is a 3-variable function in x, y, z then $\dfrac{\partial f}{\partial x}$ = derivative of f obtained by treating x as a variable and y <u>and</u> z as constants.

<u>Example</u> $f(x,y,z) = x^2 - 2y - xyz \Rightarrow$

$$\frac{\partial f}{\partial x} = 2x - yz \quad ; \quad \frac{\partial f}{\partial y} = -2 - xz \quad ; \quad \frac{\partial f}{\partial z} = -xy$$

We can also form higher partial derivatives. For example

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^3 f}{\partial z \partial y \partial x}.$$

Take partial derivative with respect to x twice

Take partial derivative with respect to y, then x

Take partial derivative with respect to x, then y then z.

<u>Example</u> $f(x,y) = x^2 + 3xy + 2y^2 \Rightarrow$

$$\frac{\partial f}{\partial x} = 2x + 3y \quad \underline{\text{and}} \quad \frac{\partial f}{\partial y} = 3x + 4y$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y \partial x} = 3, \quad \frac{\partial^2 f}{\partial y^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = 3$$

<u>Fact</u>: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ (for all functions we'll encounter)

$$\frac{\partial f}{\partial x} = -2x \quad \Rightarrow \quad \frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y} = -2y \quad \Rightarrow \quad \frac{\partial f}{\partial y}(0,0) = 0$$

As expected!

<u>Warning</u>

$$\frac{\partial f}{\partial x}(a,b) = 0 \quad \underline{\text{and}} \quad \frac{\partial f}{\partial y}(a,b) = 0 \quad \not\Rightarrow \quad f \text{ has a relative max/min at } (a,b)$$

This is exactly the same as for single variable functions.