Multivariable Functions

one input

one output

Single-variable function : $f(x) = y$

2-variable function : $f(x, y) = z$

2 inputs one output

3-variable function : $f(x, y, z) = t$

3 inputs one output

Nothing special about letters, just unknown variables

For n, any positive whole number (ie, 1, 2, 3, 4, ...), an n-variable function has n inputs and one output.

Examples

1/ (Algebraic, i.e. given by a formula)

$$f(x, y) = x^2 + y^2$$

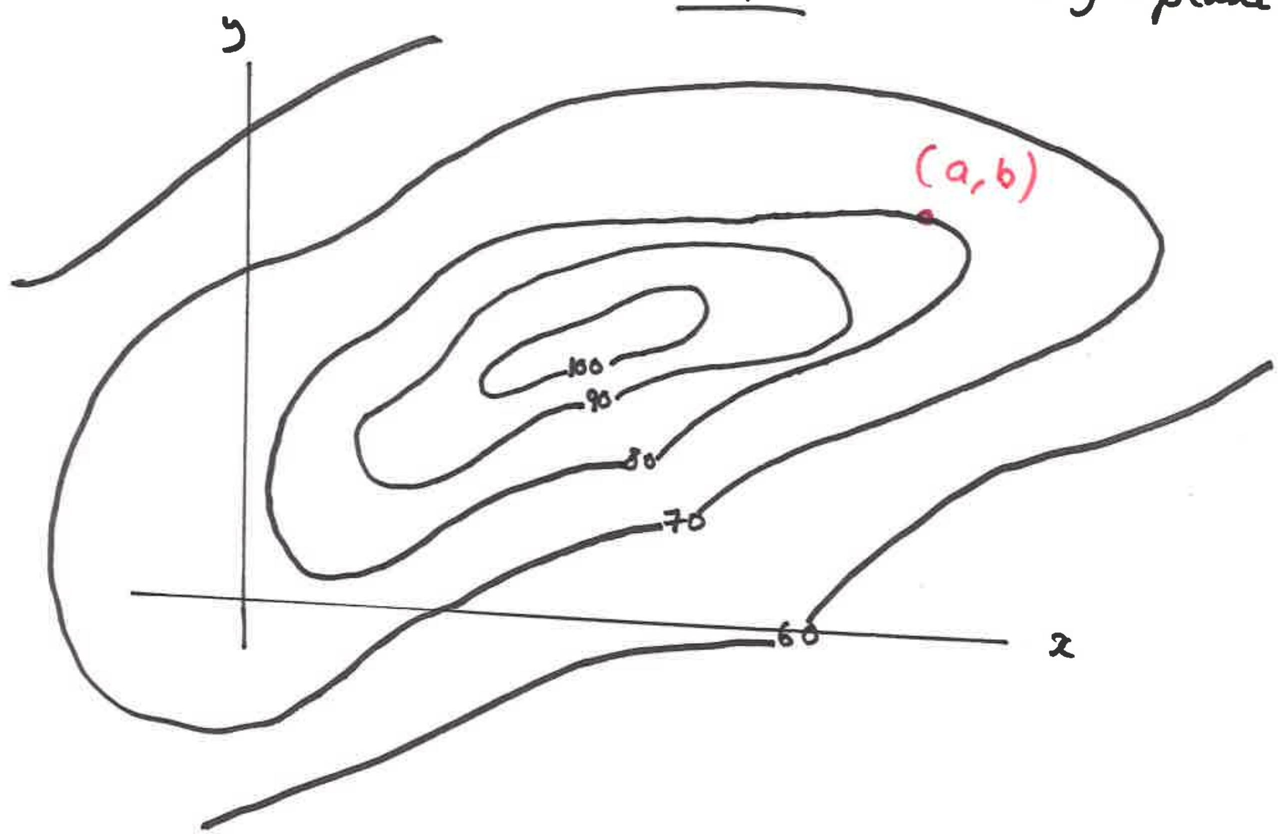
$$f(x, y, z) = 2x - 3y + 7z$$

$$f(\alpha, \beta, \gamma) = \cos(\alpha)\sin(\beta)\tan(\gamma)$$

Domain = inputs for which function makes sense.

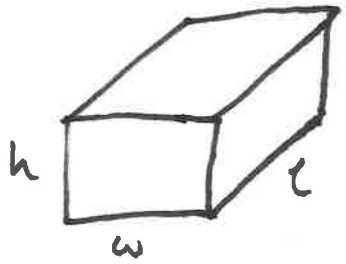
E.g. $f(x,y) = \frac{x}{y}$ has domain (x,y) with $y \neq 0$, i.e. the xy-plane minus the x-axis.

2/ (Geometric) Consider a <u>map</u> in xy-plane:



Define $f(x,y)$ = height at position (x,y)

E.g. $f(a,b) = 80$



Volume $= f(w, h, l) = whl$

3-variable function.

3/ (Physics) Kinetic Energy $= f(m, v) = \frac{1}{2}mv^2$

mass, velocity

2-variable function

4/ (Economics)

Revenue $= f(p, x) = px$

price per unit, number of units sold

Cobb-Douglas Production Equation:

Labor (ie how many workers)

Capital (ie, buildings, tools, machines, etcs)

Production (ie number of goods produced in some fixed period of time) $= f(x, y) = C\, x^{A} y^{1-A}$

(C, A fixed constants, $0 < A < 1$, $C > 0$)

E.g. $f(x,y) = 10\, x^{1/3} y^{2/3}$ or $f(x,y) = 2\, x^{1/2} y^{1/2}$

<u>Exercise</u> : Show that for <u>any</u> Cobb - Douglas equation if we <u>double</u> both labor and capital then production doubles.

(Recall : $(ab)^c = a^c b^c$ and $a^{(b+c)} = a^b \cdot a^c$)

$$f(2x, 2y) = C\,(2x)^A\,(2y)^{1-A} = C \cdot 2^A \cdot x^A \cdot 2^{1-A} \cdot y^{1-A}$$

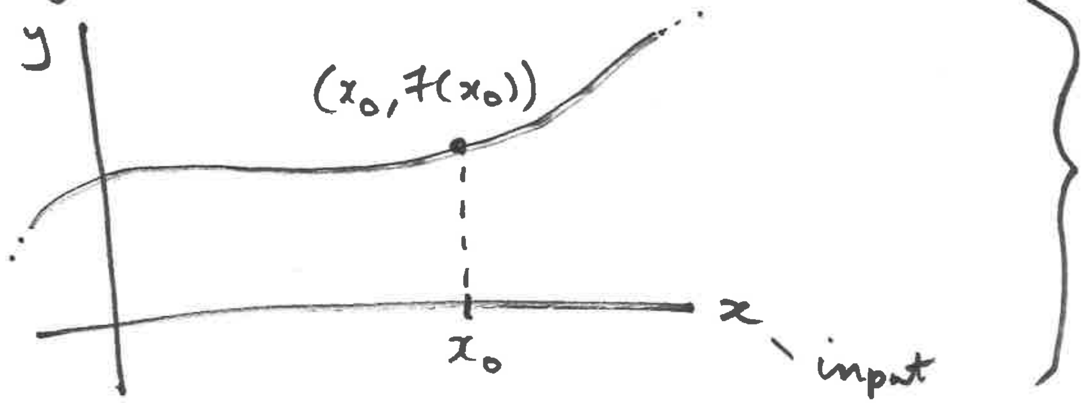
$$= C \cdot 2^A \cdot 2^{1-A} \cdot x^A y^{1-A} = 2 \cdot C \cdot x^A y^{1-A}$$

$$= 2 \cdot f(x,y) = \text{double production}$$

production after doubling labor and capital

Visualising 2 - variable Functions

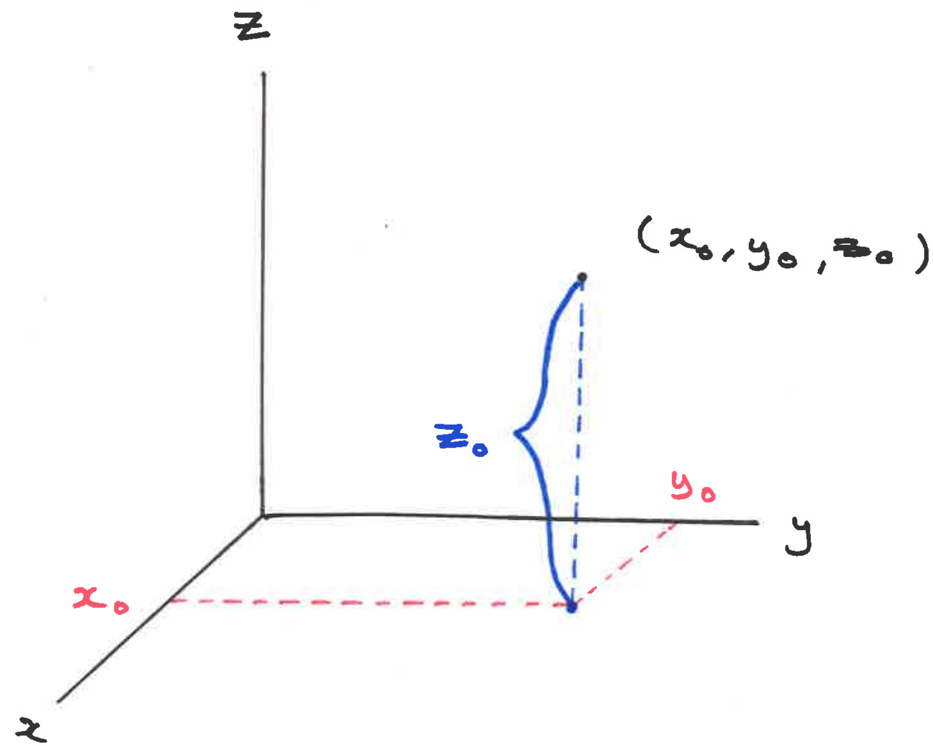
Single variable : Graph of $y = f(x)$



Need 2 dimensions as there are 2 variables x and y.

2 variables : $z = f(x,y)$

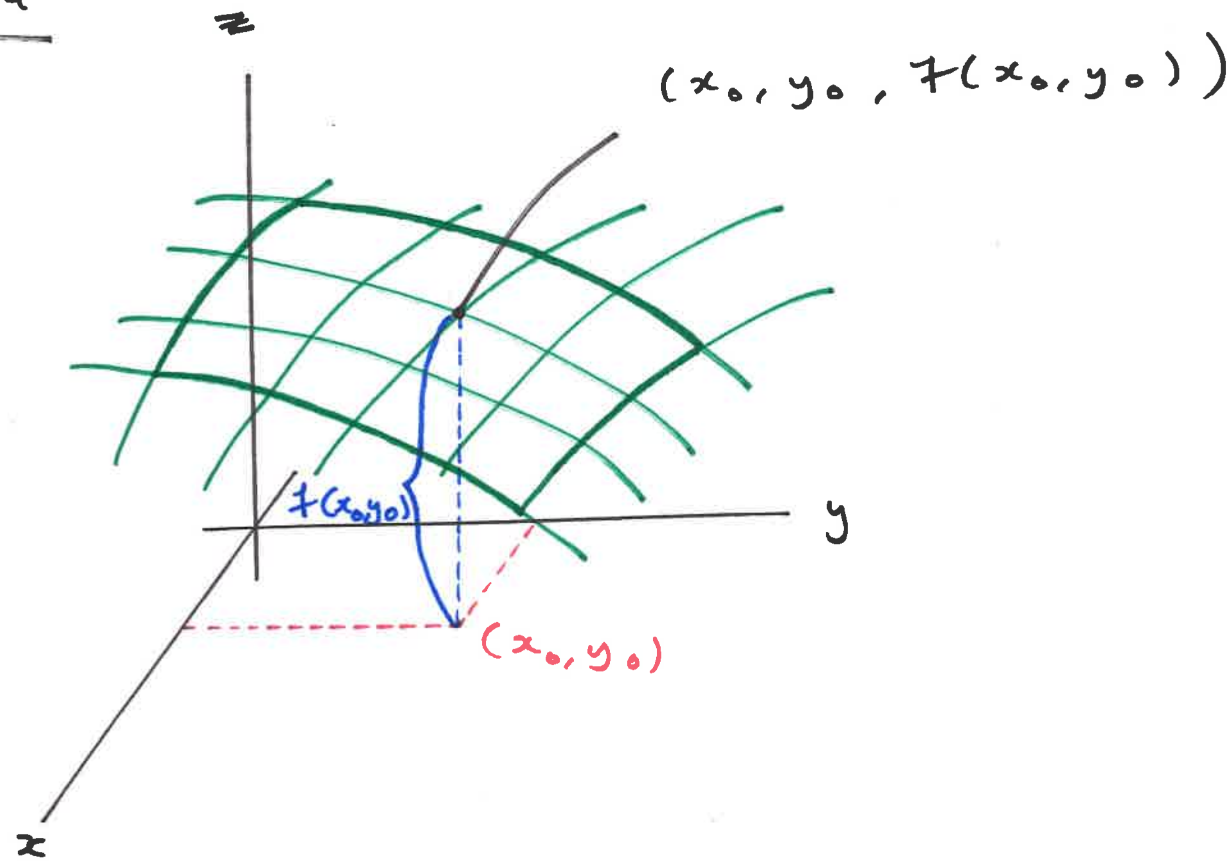
Three variables x, y, z $\Rightarrow$ Need three dimensions.



<u>Definition</u> The graph of $f(x,y)$ is all points (x_0, y_0, z_0) such that $f(x_0, y_0) = z_0$.

<u>Remark</u> : Graph of $y = f(x)$ is a <u>curve</u>.
Graph of $z = f(x,y)$ is a <u>surface</u>.

Example



Intuitively surface $z = f(x,y)$ is a landscape where $f(x,y)$ gives the height at position (x,y).

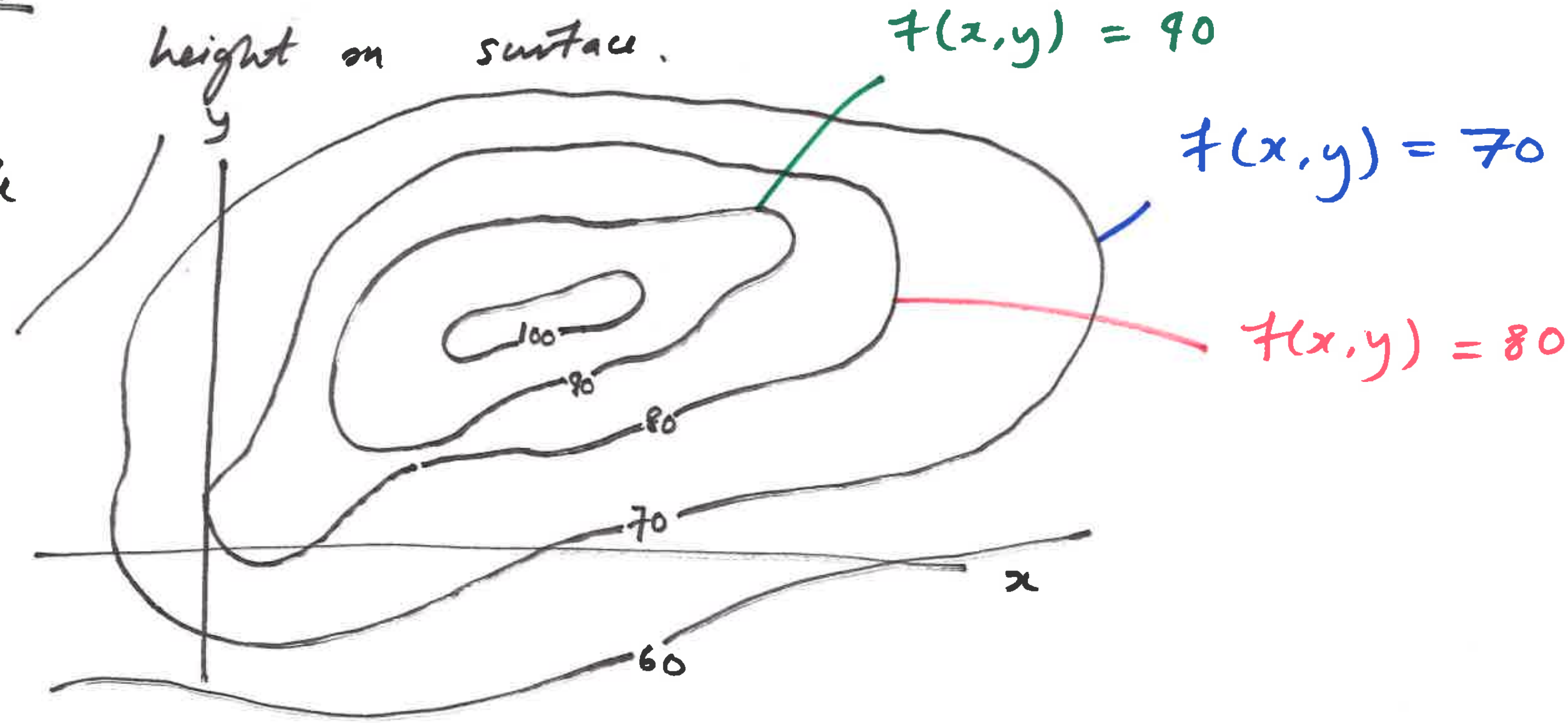
Remark In general drawing $z = f(x,y)$ is very hard. We really need a computer to do it.

Imagine now we wish to draw a <u>map</u> of our surface

$f(x,y) = z$.

<u>Strategy</u>: Draw curves in xy-plane which have same height on surface.

<u>Example</u>



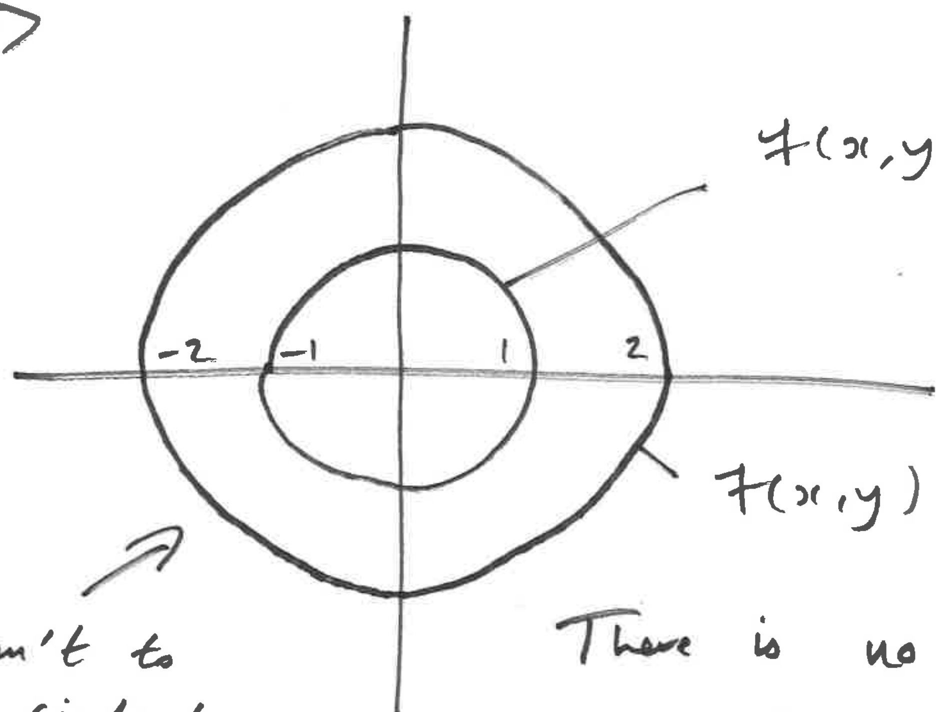
<u>Definition</u> Let $f(x,y)$ be a 2-variable function. For k a fixed number the <u>level curve</u> of height k is the curve in the xy-plane given by $f(x,y) = k$.

The level curve of height k tells us all points on surface with height k.

<u>Example</u> Draw the level curves of heights 1, 4 and -1 of the function $f(x,y) = x^2 + y^2$.

(Recall : For $r > 0$ fixed $x^2 + y^2 = r^2$ gives the circle of radius r centred at $(0,0)$)

$\Rightarrow$



$f(x,y) = x^2 + y^2 = 1 = 1^2$

$f(x,y) = x^2 + y^2 = 4 = 2^2$

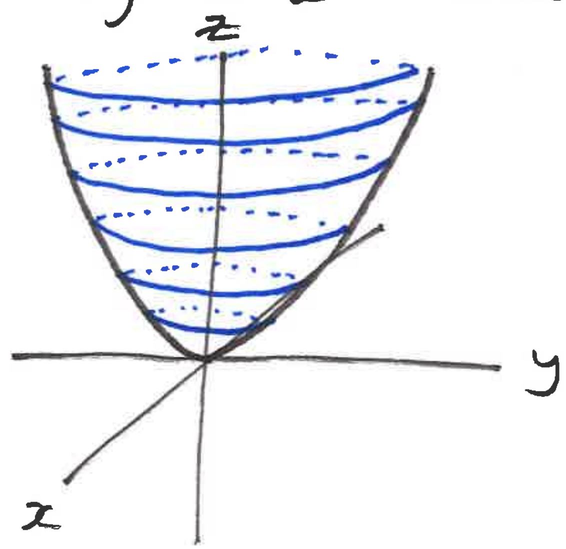
Mean't to be circles!

There is no level curve of height -1 as $x^2 + y^2 = -1$ has <u>no</u> solutions.

Drawing level curves gives us a sense of what $f(x,y) = z$ looks like. In this case

$f(x,y) = k \Rightarrow x^2+y^2 = (\sqrt{k})^2 \Rightarrow$ Level curve is circle with radius $\sqrt{k}$ and centre $(0,0)$

$\Rightarrow x^2+y^2 = z$ looks like the following:



The level curves are circles with increasing radius.

Pretty neat!

Basic Aim: Extend basic concepts from calculus (differentiation and integration) to functions of several variables to help us find maxima/minima, etc.