This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:

(a) (10 points)

$$\int \sec^2(x)\tan(x)dx.$$

Solution:

$u = \tan(x) \Rightarrow \frac{du}{dx} = \sec^2(x) \Rightarrow dx = \frac{du}{\sec^2(x)}$

$\Rightarrow \int \sec^2(x)\tan(x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2(x) + C$

(b) (10 points)

$$\int x\sin(4x)dx$$

Solution:

$f(x) = x \quad g(x) = \sin(4x)$

$f'(x) = 1 \quad G(x) = -\frac{1}{4}\cos(4x)$

$\Rightarrow \int x\sin(4x)\,dx = -\frac{1}{4}x\cos(4x) + \frac{1}{4}\int \cos(4x)\,dx$

$= \frac{-1}{4}x\cos(4x) + \frac{1}{16}\sin(4x) + C$

2. (20 points) Determine if the following improper integral is convergent or divergent:

$$\int_{-\infty}^{0} \frac{e^{-2x}}{e^{-2x}+1}dx.$$

If it is convergent determine its value.

Solution:

$$u = e^{-2x} + 1 \Rightarrow \frac{du}{dx} = -2e^{-2x} \Rightarrow dx = \frac{du}{-2e^{-2x}}$$

$$\Rightarrow \int \frac{e^{-2x}}{e^{-2x}+1} dx = -\frac{1}{2}\int \frac{1}{u}\, du = -\frac{1}{2}\ln|u| + c$$

$$= -\frac{1}{2}\ln|e^{-2x}+1| + c$$

$$\Rightarrow \int_{t}^{0} \frac{e^{-2x}}{e^{-2x}+1} dx = -\frac{1}{2}\ln|e^{-2x}+1|\Big|_{t}^{0}$$

$$= \frac{1}{2}\ln(e^{-2t}+1) - \frac{1}{2}\ln(2)$$

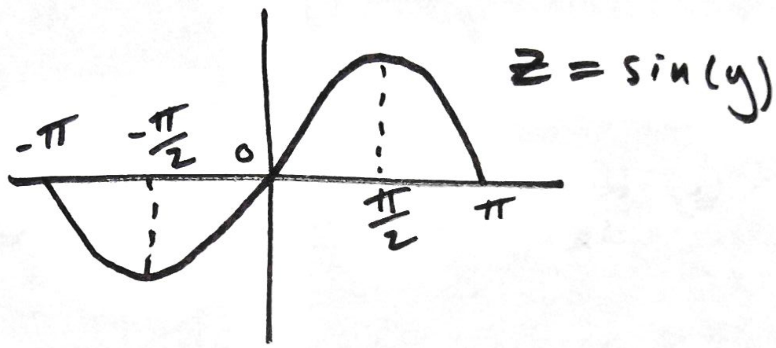
$$\lim_{t\to-\infty} e^{-2t} = \infty \Rightarrow \lim_{t\to-\infty} \ln(e^{-2t}+1) = \infty$$

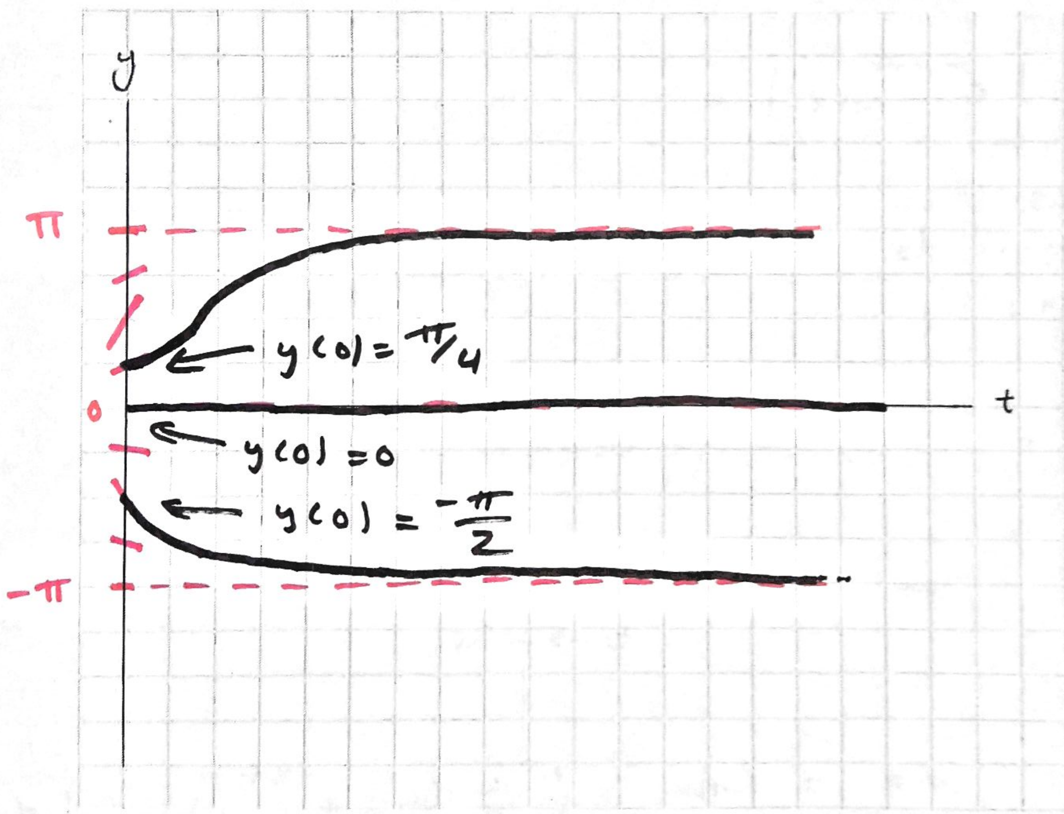
$$\Rightarrow \int_{-\infty}^{0} \frac{e^{-2x}}{e^{-2x}+1} dx = \lim_{t\to\infty}\left(\frac{1}{2}\ln(e^{-2t}+1) - \frac{1}{2}\ln(2)\right)$$

$$= \infty \Rightarrow \underline{\text{Divergent}}.$$

3. (20 points) Consider the differential equation $y' = \sin(y)$. Sketch a solution for each of the following initial conditions: $y(0) = 0$, $y(0) = -\pi/2$ and $y(0) = \pi/4$.

Solution:





4. (20 points) A population has growth constant 2. Over the time period $[0, t]$ it is expected that $1000t^2$ people will immigrate. If the population at $t = 0$ is 1000, what will it be when $t = 4$?

Solution:

Immigration over $[0,t]$ $= 1000t^2 \Rightarrow$ Immigration rate $= \frac{d(1000t^2)}{dt} = 2000t$

$y(t) =$ population at time $t \Rightarrow y' = 2y + 2000t$ (growth constant)

$\Rightarrow y' + (-2)y = 2000t \Rightarrow a(t) = -2$, $b(t) = 2000t$ $\quad A(t) = e^{-2t}$

$\Rightarrow y = \frac{1}{e^{-2t}} \int e^{-2t} \cdot 2000t \, dt = 2000 e^{2t} \int t e^{-2t} dt$

$f(t) = t \quad g(t) = e^{-2t}$

$f'(t) = 1 \quad G(t) = -\frac{1}{2} e^{-2t} \Rightarrow \int t e^{-2t} dt = -\frac{1}{2} t e^{-2t} + \frac{1}{2} \int e^{-2t} dt$

$= -\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + c$

$\Rightarrow y = 2000 e^{2t} \left(-\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + c \right)$

$= -1000t - 500 + 2000 c e^{2t}$

$y(0) = 1000 \Rightarrow -500 + 2000c = 1000 \Rightarrow 2000c = 1500$

$\Rightarrow y(t) = -1000t - 500 + 1500 e^{2t}$

$\Rightarrow y(4) = -4000 - 500 + 1500 e^8.$

5. (a) (15 points) Find a general solution to the following differential equation:

$$y' = \frac{(\ln(x))^2}{xy}$$

Solution:

$$y' = \frac{(\ln(x))^2}{x} \cdot \frac{1}{y} \leftarrow$$ never 0 so no constant solutions.

$$\int y\, dy = \int \frac{(\ln(x))^2}{x} dx$$

$$u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du \Rightarrow \int \frac{(\ln(x))^2}{x} dx$$

$$= \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln(x))^3 + C$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{3}(\ln(x))^3 + C$$

$$\Rightarrow y = \pm\sqrt{\frac{2}{3}(\ln(x))^3 + 2C} \leftarrow$$ general solution

(b) (5 points) Using part(a) find a solution which satisfies the initial condition

$$y(1) = -1.$$

Solution:

$$y(1) = -1 \Rightarrow -\sqrt{\frac{2}{3} \cdot 0^3 + 2C} = -1 \Rightarrow 2C = 1$$

$$\Rightarrow y = -\sqrt{\frac{2}{3}(\ln(x))^3 + 1}$$

END OF EXAM