This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:
   (a) (10 points)

$$\int x \sec^2(3x^2)dx$$

Solution:

$u = 3x^2 \Rightarrow \frac{du}{dx} = 6x \Rightarrow dx = \frac{du}{6x} \Rightarrow$

$\int x \sec^2(3x^2)\,dx = \frac{1}{6}\int \sec^2(u)\,du = \frac{1}{6}\tan(u) + C$

$= \frac{1}{6}\tan(3x^2) + C$

(b) (10 points)

$$\int x^2 \ln(x^4)dx.$$

Solution:

$x^2 \ln(x^4) = 4x^2 \ln(x)$

$f(x) = \ln(x) \quad g(x) = 4x^2$

$f'(x) = \frac{1}{x} \quad G(x) = \frac{4}{3}x^3 \quad \Rightarrow \int x^2 \ln(x^4)\,dx = \frac{4}{3}x^3 \ln(x) - \int \frac{4}{3}x^2\,dx$

$= \frac{4}{3}x^3 \ln(x) - \frac{4}{9}x^3 + C$

2. (25 points) Find a general solution to the following differential equation:

$$xy' + 4y = e^{x^4}$$

Solution:

$xy' + 4y = e^{x^4} \Rightarrow y' + \frac{4}{x}y = \frac{e^{x^4}}{x} \Rightarrow$ $a(x) = \frac{4}{x}$, $b(x) = \frac{e^{x^4}}{x}$

Let $A(x) = 4\ln|x| = \ln(x^4) \Rightarrow e^{A(x)} = x^4$

$\Rightarrow y = \frac{1}{x^4}\int x^4 \cdot \frac{e^{x^4}}{x}\,dx = \frac{1}{x^4}\int x^3 e^{x^4}\,dx$

$u = x^4 \Rightarrow \frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3} \Rightarrow$

$\int x^3 e^{x^4}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$

$\Rightarrow y = \frac{e^{x^4}}{4x^4} + \frac{C}{x^4}$ is general solution

3. (20 points) The population density at distance t miles form the centre of a city is $1000\cos(t/2)$ people per square kilometer. How many people live between 1 and 2 km of the city center? You do not need to simplify your answer.

Solution:

Population between 1 and 2 km from center $= \int_1^2 1000\cos(t/2)\cdot 2\pi t\,dt$

$= 2000\pi \int_1^2 t\cos(t/2)\,dt$

$f(t) = t \quad g(t) = \cos(t/2)$

$f'(t) = 1 \quad G(t) = 2\sin(t/2)$

$\Rightarrow \int t\cos(t/2)\,dt = 2t\sin(t/2) - 2\int \sin(t/2)\,dt$

$= 2t\sin(t/2) + 4\cos(t/2) + C$

$\Rightarrow \int_1^2 t\cos(t/2)\,dt = 2t\sin(t/2) + 4\cos(t/2)\Big|_1^2$

$= 4\sin(1) + 4\cos(1) - 2\sin(\frac{1}{2}) - 4\cos(1/2)$

$\Rightarrow$ Population $= 2000\pi\left(4\sin(1) + 4\cos(1) - 2\sin(1/2) - 4\cos(1/2)\right)$

4. (a) (10 points) Find a general solution to the following differential equation:

$$y' = te^t(y-1)$$

Solution:

$y - 1 = 0 \Rightarrow y = 1 \Rightarrow y = 1$ is only constant solution.

$$\int \frac{1}{y-1}\,dy = \int te^t\,dt$$

$f(t) = t$, $g(t) = e^t$

$f'(t) = 1$, $G(t) = e^t$ $\Rightarrow \int te^t\,dt = te^t - e^t + C$

$\Rightarrow \ln|y-1| = te^t - e^t + C$

$\Rightarrow y = \pm e^C \cdot e^{(te^t - e^t)} + 1$

$\Rightarrow y = \begin{cases} 1 \\ \pm e^C \cdot e^{(te^t - e^t)} + 1 \quad (C \text{ arbitrary}) \end{cases}$

same as $y = Ae^{(te^t - e^t)} + 1$ for A arbitrary is general solution.

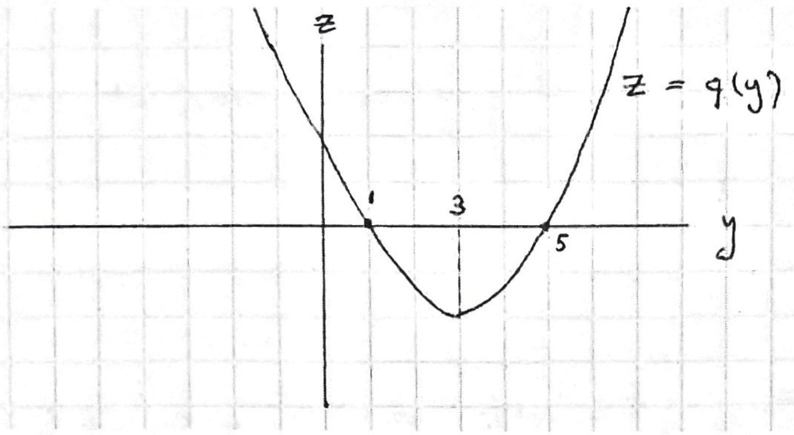
(b) (5 points) Using part(a) find a solution which satisfies the initial condition

$$y(1) = 1.$$

Solution:

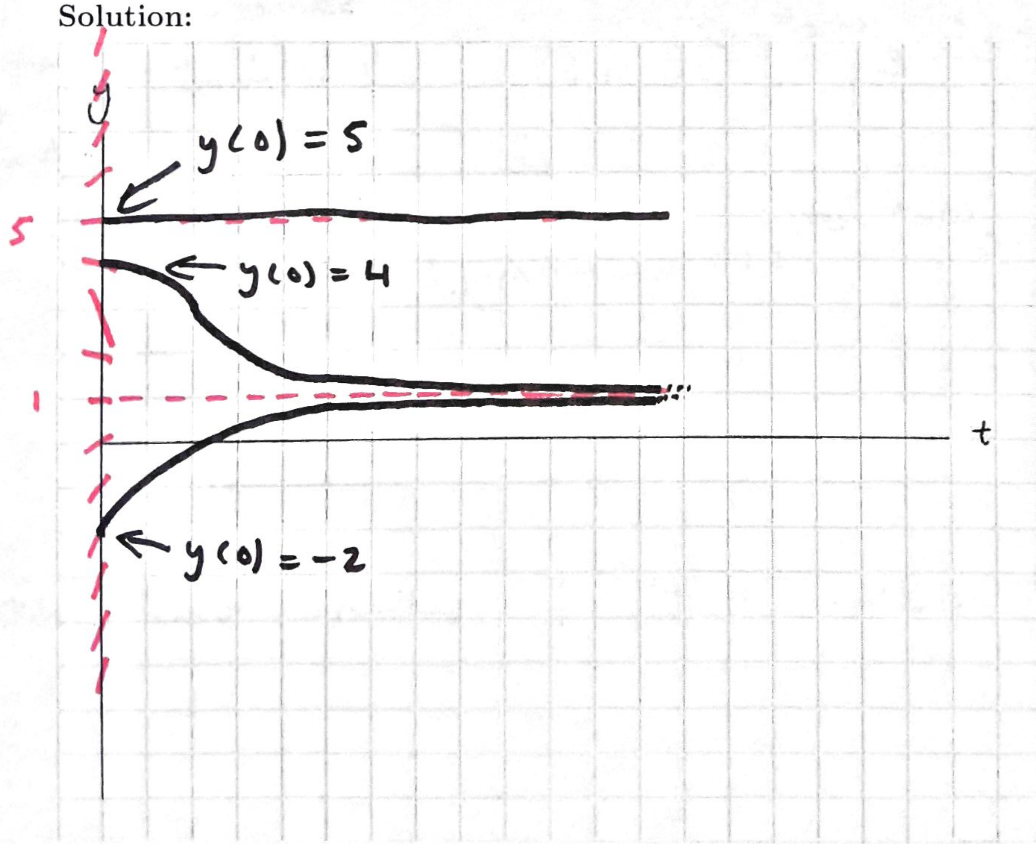
The constant solution $y(t) = 1$ satisfies this condition.

5. (20 points) Consider the differential equation of the form $y' = q(y)$, where the graph $z = q(y)$ is as follows:



Sketch a solution for each of the following initial conditions: $y(0) = -2$, $y(0) = 5$ and $y(0) = 4$.

Solution:



END OF EXAM