This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Find all first partial derivatives of the following functions:

(a) (10 points)

$$f(x, y) = \ln(x^4 + 9y^2)$$

Solution:

$$\frac{\partial f}{\partial x} = \frac{4x^3}{x^4 + 9y^2}$$

$$\frac{\partial f}{\partial y} = \frac{18y}{x^4 + 9y^2}$$

(b) (10 points)

$$f(x, y, z) = \frac{\cos(xyz)}{x+1}$$

Solution:

$$\frac{\partial f}{\partial x} = \frac{-yz\sin(xyz)(x+1) - \cos(xyz)}{(x+1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-xz\sin(xyz)}{x+1}$$

$$\frac{\partial f}{\partial z} = \frac{-xy\sin(xyz)}{x+1}$$

2. (25 points) Calculate the following double integral

$$\iint_R \frac{3+5y}{\sqrt{x}}\,dxdy,$$

where R is the region given by $4 \le x \le 9$ and $1 \le y \le 2$.

Solution:

$$\iint_R \frac{3+5y}{\sqrt{x}}\,dx\,dy = \int_1^2 \left(\int_4^9 (3+5y)\, x^{-\frac{1}{2}}\, dx \right) dy$$

$$= \int_1^2 (3+5y)\, 2\sqrt{x} \Big|_4^9 \, dy$$

$$= \int_1^2 2(3+5y)\, dy$$

$$= 6y + 5y^2 \Big|_1^2 = 12 + 20 - 6 - 5$$

$$= 21$$

3. Let $f(x, y) = e^{x(y+1)}$

(a) (10 points) Find all the possible relative maxima/minima using the first derivative test.

Solution:

$$\frac{\partial f}{\partial x} = (y+1)\, e^{x(y+1)} = 0$$

$$\frac{\partial f}{\partial y} = x\, e^{x(y+1)} = 0$$

$$\Rightarrow \begin{cases} y+1 = 0 \\ x = 0 \end{cases} \Rightarrow y = -1$$

$\Rightarrow$ $(0,-1)$ only possible location for max/min.

(b) (10 points) Use the second derivative test to determine the nature of each such point.

Solution:

$$\frac{\partial^2 f}{\partial x^2} = (y+1)^2 e^{x(y+1)}, \quad \frac{\partial^2 f}{\partial y^2} = x^2 e^{x(y+1)}$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{x(y+1)} + x(y+1)\, e^{x(y+1)}$$

$$\Rightarrow D(0,-1) = 0 \cdot 0 - (1)^2 = -1$$

$\Rightarrow$ $(0,-1)$ saddle point.

4. (25 points) Using the method of Lagrange Multipliers, find three positive numbers whose sum is 30 and whose product is maximized. You may assume a maximum exists without justification.

Solution:

Objective : xyz

Constraint : $x+y+z-30=0 \quad (x,y,z>0)$

$$F(x,y,z,\lambda) = xyz + \lambda x + \lambda y + \lambda z - 30\lambda$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= yz + \lambda = 0 \\ \frac{\partial F}{\partial y} &= xz + \lambda = 0 \\ \frac{\partial F}{\partial z} &= xy + \lambda = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} yz &= xz \\ xz &= xy \end{aligned} \Rightarrow \begin{aligned} y &= x \\ z &= y \end{aligned}$$

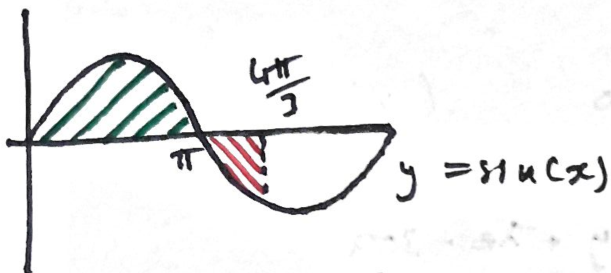
$$\frac{\partial F}{\partial \lambda} = x+y+z-30=0 \Rightarrow 3x-30=0 \Rightarrow x=10$$

$$\Rightarrow y=10 \Rightarrow z=10$$

$\Rightarrow$ Max occurs when $x=0,\ y=0,\ z=0$

5. (20 points) Determine the total area enclosed by the graph $y = \sin(x)$ and the x-axis between $x = 0$ and $x = 4\pi/3$. Note that I am asking for the total area, namely the sum of the areas above and below the x-axis.

Solution:

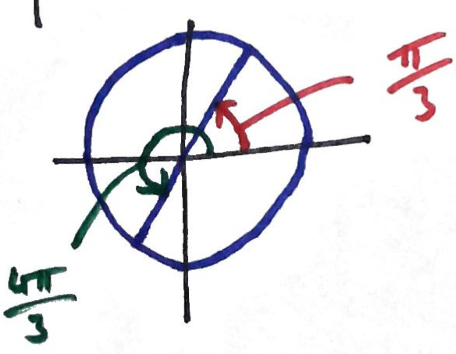


$$\text{Area}(\text{green}) = \int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = -\cos(\pi) - (-\cos(0))$$
$$= -(-1) - (-1)$$
$$= 2$$

$$\text{Area}(\text{red}) = \int_{\pi}^{\frac{4\pi}{3}} -\sin(x)\,dx = \cos(x)\Big|_{\pi}^{\frac{4\pi}{3}}$$

$$= \cos\left(\frac{4\pi}{3}\right) - \cos(\pi) = \frac{-1}{2} - (-1) = \frac{1}{2}$$

$\Rightarrow$ Total area enclosed $= 2 + \frac{1}{2} = \frac{5}{2}$



$\Rightarrow \cos\left(\frac{\pi}{3}\right) = -\cos\left(\frac{4\pi}{3}\right)$

$\Rightarrow \cos\left(\frac{4\pi}{3}\right) = \frac{-1}{2}$

END OF EXAM