This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Find all first partial derivatives of the following functions:

(a) (5 points)

$$f(x, y) = \tan(y + 2x).$$

Solution:

$$\frac{\partial f}{\partial x} = 2\sec^2(y+2x)$$

$$\frac{\partial f}{\partial y} = \sec^2(y+2x)$$

(b) (15 points)

$$f(x, y, z) = ye^{(xy+z)}.$$

Solution:

$$\frac{\partial f}{\partial x} = y^2 e^{(xy+z)}$$

$$\frac{\partial f}{\partial y} = e^{(xy+z)} + xy\, e^{(xy+z)}$$

$$\frac{\partial f}{\partial z} = y\, e^{(xy+z)}$$

2. Let $f(x, y) = x^2 + xy + y^2 - 6x - 3$.

(a) (10 points) Find all the possible relative maxima/minima using the first derivative test.

Solution:

$$\frac{\partial f}{\partial x} = 2x + y - 6 = 0$$

$$\frac{\partial f}{\partial y} = x + 2y = 0 \Rightarrow x = -2y \Rightarrow -4y + y - 6 = 0$$

$$\Rightarrow y = -2 \Rightarrow x = 4$$

$\Rightarrow (4, -2)$ only potential max/min

(b) (10 points) Use the second derivative test to determine the nature of each such point.

Solution:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 1 \Rightarrow D(x, y) = 3$$

$\Rightarrow D(4, -2) = 3 > 0$ and $\frac{\partial^2 f}{\partial x^2}(4, -2) = 2 > 0 \Rightarrow$

$(4, -2)$ gives a relative min.

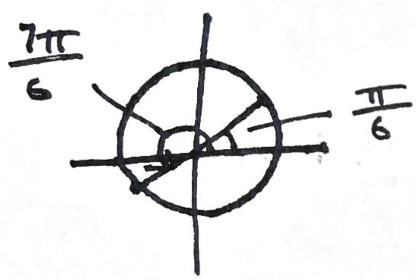
3. (20 points) Determine the area under the graph $y = \cos(x) + 1$ between $x = -\pi/4$ and $x = \frac{7\pi}{6}$. Simplify your answer as much as possible.

Solution:

$\cos(x) + 1 \geq 0$ for all $x \Rightarrow$ Area under graph is

$$\int_{-\pi/4}^{\frac{7\pi}{6}} (\cos(x) + 1)\,dx = \sin(x) + x \Big|_{\frac{-\pi}{4}}^{\frac{7\pi}{6}}$$

$$= \sin\left(\frac{7\pi}{6}\right) + \frac{7\pi}{6} - \sin\left(-\frac{\pi}{4}\right) - \left(-\frac{\pi}{4}\right)$$



$\Rightarrow \sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = \frac{-1}{2}$

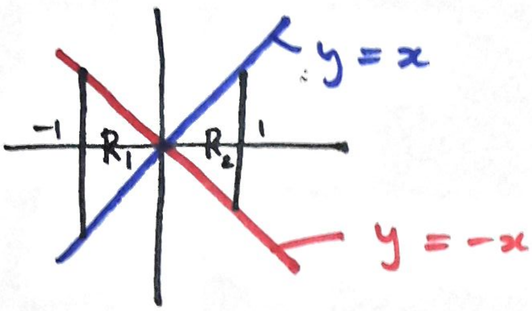
$$\Rightarrow \text{Area} = \frac{-1}{2} + \frac{7\pi}{6} + \frac{1}{\sqrt{2}} + \frac{\pi}{4}$$

4. (25 points) Calculate the following double integral

$$\iint_R x^2y^2dxdy,$$

where R is the region enclosed by the lines $y = x$ and $y = -x$ between $x = -1$ and $x = 1$.

Solution:



$$\iint_{R_1} x^2y^2\,dx\,dy = \int_0^1 \left(\int_{-x}^{x} x^2y^2\,dy \right) dx = \int_0^1 \left. \frac{x^2}{3} y^3 \right|_{-x}^{x} dx$$

$$= \int_0^1 \frac{2x^5}{3}\,dx = \left. \frac{2}{18} x^6 \right|_0^1 = \frac{1}{9}$$

$$\iint_{R_2} x^2y^2\,dx\,dy = \int_{-1}^{0} \left(\int_{x}^{-x} x^2y^2\,dy \right) dx = \int_{-1}^{0} \left(\left. \frac{x^2}{3} y^3 \right|_{x}^{-x} \right) dx$$

$$= \int_{-1}^{0} \frac{-2x^5}{3}\,dx = \left. \frac{-2}{18} x^6 \right|_{-1}^{0} = \frac{1}{9}$$

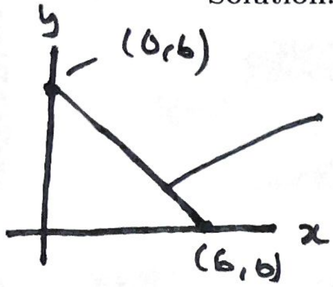
$$\Rightarrow \iint_R x^2y^2\,dx\,dy = \frac{2}{9}.$$

5. (25 points) The profits from the sale of x units of radiators for automobiles and y units of radiators for generators is given by

$$P(x, y) = -x^2 - y^2 + 4x + 8y.$$

Find the values of x and y that lead to the maximum profit if the firm must produce 6 units of radiators. Use the method of Lagrange Multipliers to solve this problem. Be sure to justify why it is a maximum.

Solution:



$x + y - 6 = 0$

$F(x, y, \lambda) = -x^2 - y^2 + 4x + 8y + \lambda x + \lambda y - 6\lambda$

$\frac{\partial F}{\partial x} = -2x + 4 + \lambda = 0$ $\qquad \lambda = 2x - 4$

$\frac{\partial F}{\partial y} = -2y + 8 + \lambda = 0 \Rightarrow \lambda = 2y - 8 \Rightarrow 2x - 4 = 2y - 8$

$\frac{\partial F}{\partial \lambda} = x + y - 6 = 0 \Rightarrow x = y - 2$

$\Rightarrow y - 2 + y - 6 = 0 \Rightarrow 2y - 8 = 0 \Rightarrow y = 4 \Rightarrow x = 2$

$\Rightarrow (2, 4)$ only Lagrange point.

$P(2,4) = 20$

$P(6,0) = -12 \qquad \Rightarrow x = 2,\ y = 4$ gives max profit.

$P(0,6) = 12$

END OF EXAM