This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Find all first partial derivatives of the following functions:

   (a) (5 points)

$$f(x, y) = \sin(x^2 y).$$

Solution:

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2y)$$

$$\frac{\partial f}{\partial y} = x^2\cos(x^2y)$$

   (b) (15 points)

$$f(x, y, z) = zy\ln(xy + 2).$$

Solution:

$$\frac{\partial f}{\partial x} = \frac{zy^2}{xy+2}$$

$$\frac{\partial f}{\partial y} = z\ln(xy+2) + \frac{xyz}{xy+2}$$

$$\frac{\partial f}{\partial z} = y\ln(xy+2)$$

2. Let $f(x, y) = x^2 + xy - 2x - 2y + 2$

   (a) (15 points) Find all the possible relative maxima/minima using the first derivative test.

   Solution:

$$\frac{\partial f}{\partial x} = 2x + y - 2 = 0$$
$$\frac{\partial f}{\partial y} = x - 2 = 0$$
$$\Rightarrow \quad x = 2 \Rightarrow y = y = -2$$

$(2,-2)$ is only possible location of relative max/min

   (b) (10 points) Use the second derivative test to determine the nature of each such point.

   Solution:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = 1 \Rightarrow D(x,y) = -1$$

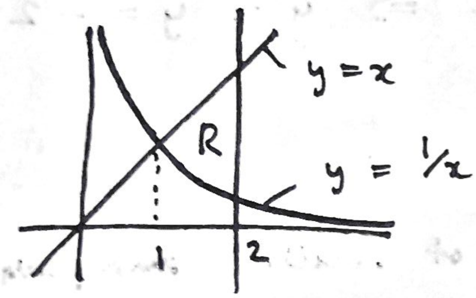
$$\Rightarrow D(2,-2) < 0 \Rightarrow (2,-2) \text{ saddle point}$$

3. (20 points) Calculate the following double integral

$$\iint_R (x+y^2)\,dx\,dy,$$

where R is the region enclosed by $y=x$, $y=1/x$ and $x=2$.

Solution:



$$\int_{1/x}^{x} (x+y^2)\,dy = xy + \frac{y^3}{3}\Big|_{1/x}^{x} = x^2 + \frac{x^3}{3} - 1 - \frac{x^{-3}}{3}$$

$$\Rightarrow \iint_R x+y^2\,dx\,dy = \int_1^2 x^2 + \frac{x^3}{3} - 1 - \frac{x^{-3}}{3}\,dx$$

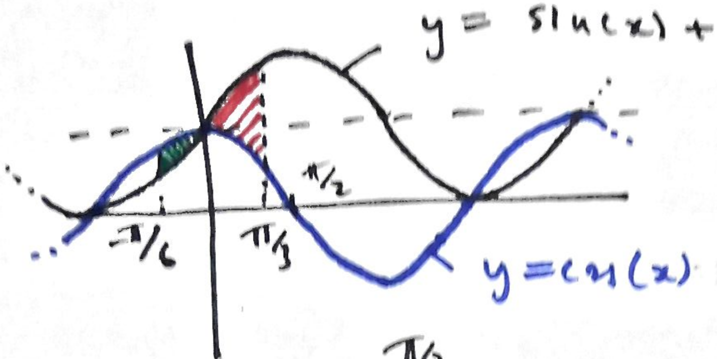
$$= \frac{x^3}{3} + \frac{x^4}{12} - x + \frac{x^{-2}}{6}\Big|_1^2$$

$$= \left(\frac{8}{3} + \frac{16}{12} - 2 + \frac{1}{24}\right) - \left(\frac{1}{3} + \frac{1}{12} - 1 + \frac{1}{6}\right)$$

$$= \frac{54}{24}$$

4. (20 points) Determine the area enclosed by the graph $y = \sin(x) + 1$ and $y = \cos(x)$ between $x = -\pi/6$ and $x = \pi/3$. Simplify your answer as much as possible.

Solution:



$$\text{Area (red)} = \int_0^{\pi/3} \sin(x) + 1 - \cos(x)\, dx = -\cos(x) + x - \sin(x) \Big|_0^{\pi/3}$$

$$= \left(-\cos(\pi/3) + \pi/3 - \sin(\pi/3)\right) - \left(-\cos(0) + 0 - \sin(0)\right)$$

$$= -\frac{1}{2} + \pi/3 - \frac{\sqrt{3}}{2} + 1 = \frac{1}{2} - \frac{\sqrt{3}}{2} + \pi/3$$

$$\text{Area (green)} = \int_{-\frac{\pi}{6}}^{0} \cos(x) - \sin(x) - 1\, dx = \sin(x) + \cos(x) - x \Big|_{-\frac{\pi}{6}}^{0}$$

$$= (\sin(0) + \cos(0) - 0) - \left(\sin\left(\frac{-\pi}{6}\right) + \cos\left(\frac{-\pi}{6}\right) - \left(\frac{-\pi}{6}\right)\right)$$

$$= 1 + \frac{1}{2} - \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \frac{3 - \sqrt{3}}{2} - \frac{\pi}{6}$$

$$\Rightarrow \text{Area enclosed} = 2 - \sqrt{3} + \frac{\pi}{6}.$$

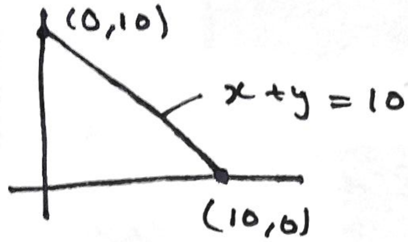
5. (25 points) The total cost to produce x large jewellery-making kits and y small ones is given by

$$C(x, y) = 2x^2 + 6y^2 + 4xy + 10.$$

If a total of ten kits must be made, how should production be allocated so that total cost is minimized? Use the method of Lagrange Multipliers to solve this problem. Be sure to justify why it is a minimum.

Solution:

$$x + y = 10$$
$$\Downarrow$$
$$x + y - 10 = 0$$



$$F(x, y, \lambda) = 2x^2 + 6y^2 + 4xy + 10 + \lambda x + \lambda y - 10\lambda$$

$$\Rightarrow \frac{\partial F}{\partial x} = 4x + 4y + \lambda = 0$$

$$\frac{\partial F}{\partial y} = 12y + 4x + \lambda = 0 \Rightarrow \begin{array}{l} \lambda = -4x - 4y \\ \lambda = -12y - 4x \end{array}$$

$$\frac{\partial F}{\partial \lambda} = x + y - 10 = 0$$

$$\Rightarrow -4x - 4y = -12y - 4x \Rightarrow 8y = 0 \Rightarrow y = 0$$

$\Rightarrow x = 10 \Rightarrow (10, 0)$ potential max/min

Check end points : $C(10, 0) = 200$, $C(0, 10) = 600$ $\Rightarrow$ $x = 10$, $y = 0$ minimizes cost.

END OF EXAM