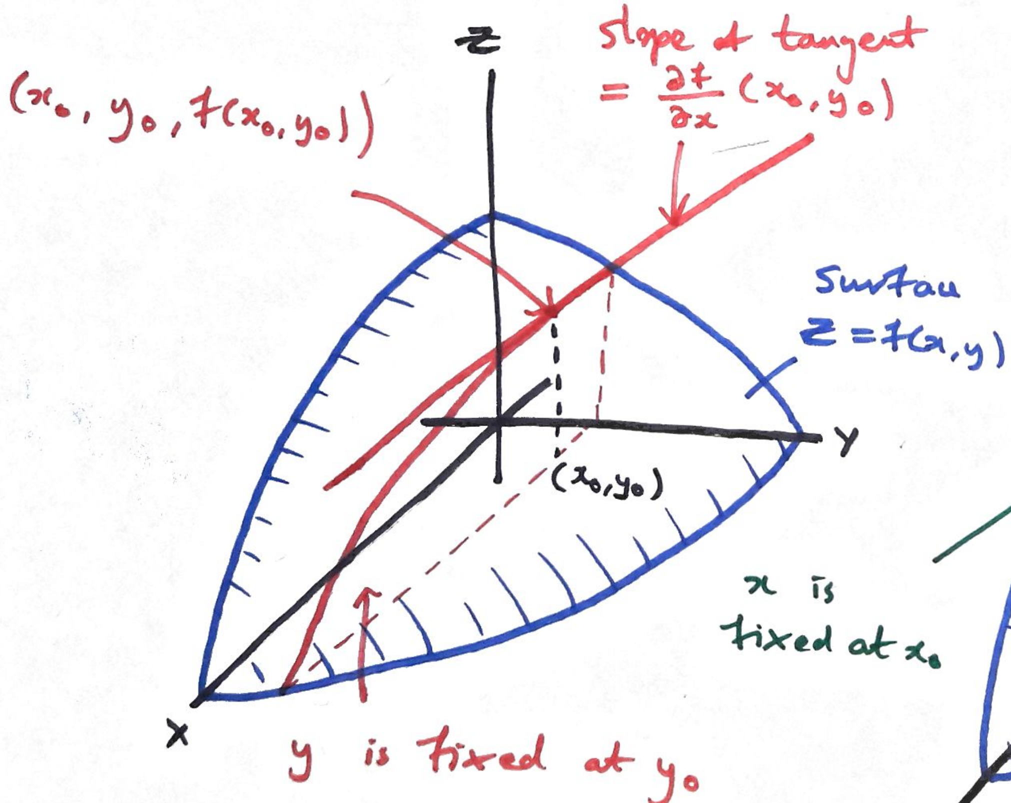
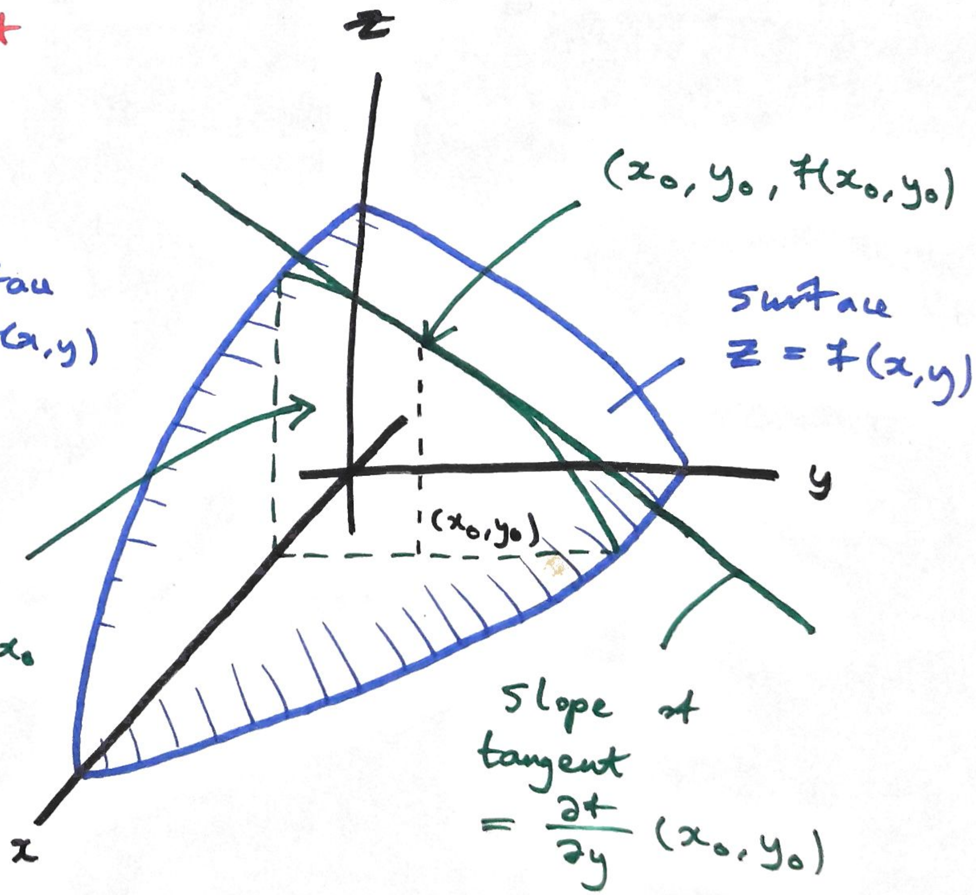
Maxima and Minima of Multivariable Functions

$f(x,y)$ - 2-variable function

Q: Can we geometrically interpret $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$?

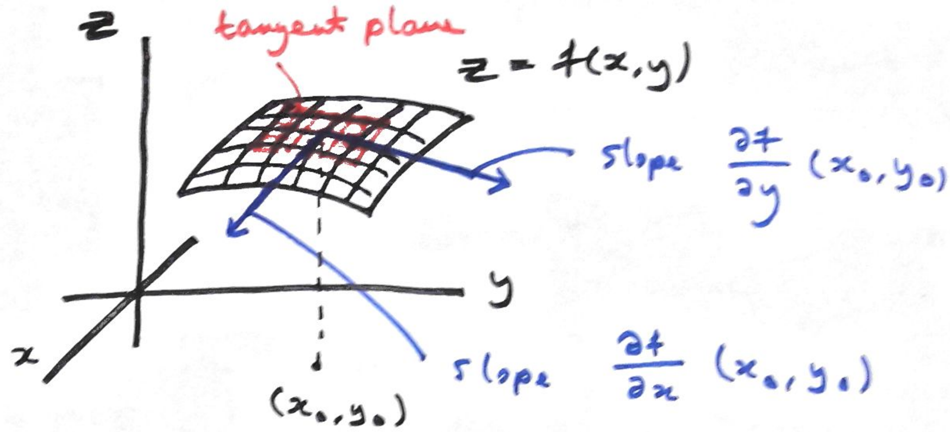




Intuitively: At $(x_0, y_0, f(x_0,y_0))$ on the surface $z = f(x,y)$ there is a <u>tangent plane</u>.

$\frac{\partial f}{\partial x}(x_0, y_0)$ = slope in x-direction

$\frac{\partial f}{\partial y}(x_0, y_0)$ = slope in y-direction

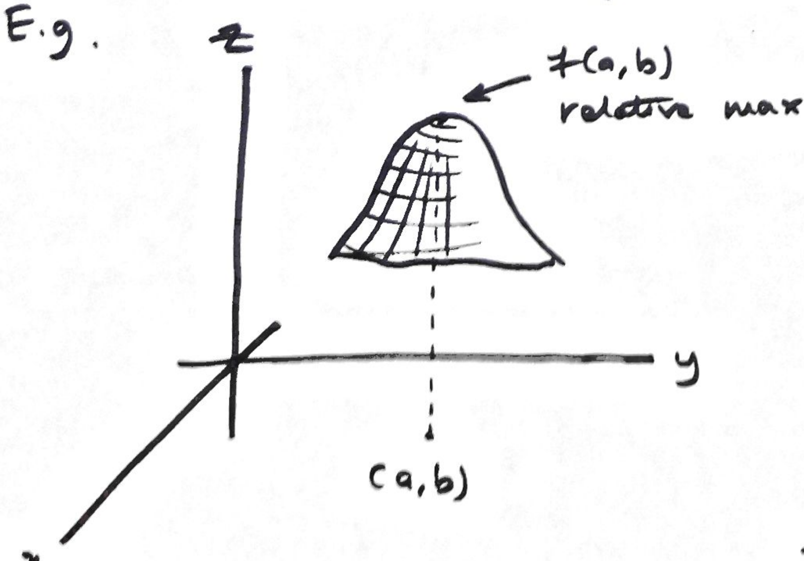


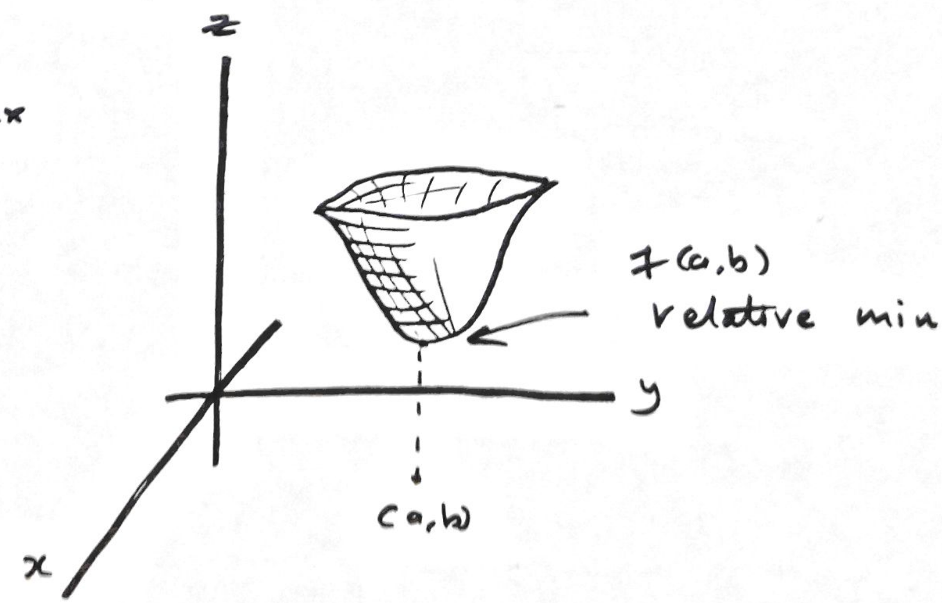
Definition

1/ We say f has a <u>relative max</u> at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) sufficiently near (a,b)

2/ We say f has a <u>relative min</u> at (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) sufficiently near (a,b)

E.g.

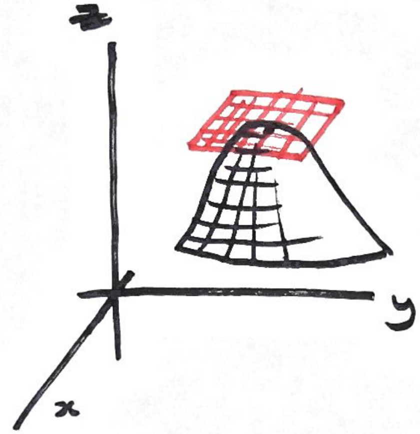




1st Derivative Test

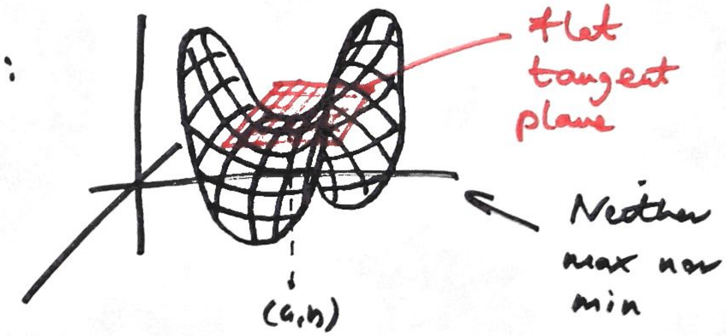
f has relative max / min at (a,b) $\Rightarrow$ Tangent plane at $(a, b, f(a,b))$ is flat (ie parallel to xy-plane)

$$\Rightarrow \frac{\partial f}{\partial x}(a,b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b) = 0$$



Remark If $\frac{\partial f}{\partial x}(a,b) = 0 = \frac{\partial f}{\partial y}(a,b)$ we say (a,b) is a critical point. So f has a rel. max/min at (a,b) $\Rightarrow$ (a,b) critical point.

Converse is false. E.g. A saddle point :



Same as inflection points in single-variable case.

Definition $D(x,y)$, the "discriminant" of f is the function

$$D(x,y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

2nd Derivative Test

Suppose (a,b) is a critical point of f $\left(\frac{\partial f}{\partial x}(a,b) = 0 = \frac{\partial f}{\partial y}(a,b) \right)$

1/ $D(a,b) > 0$ <u>and</u> $\frac{\partial^2 f}{\partial x^2}(a,b) < 0 \Rightarrow f(a,b)$ gives <u>relative max</u>

2/ $D(a,b) > 0$ <u>and</u> $\frac{\partial^2 f}{\partial x^2}(a,b) > 0 \Rightarrow f(a,b)$ gives <u>relative min</u>

3/ $D(a,b) < 0 \Rightarrow$ Saddle point

4/ $D(a,b) = 0 \Rightarrow$ Inconclusive

Strategy to Find Relative Max / Min :

A/ Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and find critical points by solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

B/ Calculate $\frac{\partial^2 f}{\partial x^2}$ and $D(x,y)$ and evaluate at critical points. Draw conclusion based on 2nd Derivative test.

Example

1/ $f(x,y) = 2 - x^2 - y^2 \Rightarrow \frac{\partial f}{\partial x} = -2x$; $\frac{\partial f}{\partial y} = -2y$

$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} \Rightarrow -2x = 0 = -2y \Rightarrow x = 0 = y \Rightarrow$

$(0,0)$ only critical point.

$\frac{\partial^2 f}{\partial x^2} = -2$, $\frac{\partial^2 f}{\partial y^2} = -2$, $\frac{\partial^2 f}{\partial x \partial y} = 0 \Rightarrow D(x,y) = 4$

$\Rightarrow D(0,0) = 4 > 0$ <u>and</u> $\frac{\partial^2 f}{\partial x^2}(0,0) = -2 < 0 \Rightarrow$ f has rel. max at $(0,0)$

2/ $f(x,y) = xy \Rightarrow \frac{\partial f}{\partial x} = y$; $\frac{\partial f}{\partial y} = x$, $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} \Rightarrow y = 0 = x$

$\Rightarrow (0,0)$ only critical point

$\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial x \partial y} = 1 \Rightarrow D(x,y) = -1$

$\Rightarrow D(0,0) < 0 \Rightarrow$ f has saddle point at $(0,0)$

3/ $f(x,y) = x^3 - y^2 - 12x + 6y \Rightarrow \frac{\partial f}{\partial x} = 3x^2 - 12$; $\frac{\partial f}{\partial y} = -2y + 6$

$3x^2 - 12 = 0 \Rightarrow x = \pm 2$, $-2y + 6 = 0 \Rightarrow y = 3$

$\Rightarrow (2,3)$ and $(-2,3)$ critical points

$\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial y^2} = -2$, $\frac{\partial^2 f}{\partial x \partial y} = 0 \Rightarrow D(x,y) = -12x$

$D(2,3) = -24 < 0 \Rightarrow$ Saddle point at $(2,3)$

$D(-2,3) = 24 > 0$, $\frac{\partial^2 f}{\partial x^2}(-2,3) = -12 < 0 \Rightarrow$ Relative max at $(-2,3)$