Linear 1st-Order Differential Equations

$$\frac{dy}{dx} + a(x)\, y = b(x)$$ — Linear 1st-order Diff. Eq.

Examples: $y' + xy = x^2+1$

$xy' + y = 2\sin(x) \Rightarrow y' + \frac{1}{x}\cdot y = \frac{2\sin(x)}{x}$

Aim: Find general solution.

Let $A(x)$ be an antiderivative of $a(x)$, ie $\int a(x)\,dx = A(x) + C$.

Define $I(x) = e^{A(x)}$, called the "integrating factor".

Example $y' + \frac{1}{x}\cdot y = \frac{2\sin(x)}{x}$ (assume $x > 0$)

$\Rightarrow a(x) = \frac{1}{x}$. Let $A(x) = \ln(x) \Rightarrow I(x) = e^{\ln(x)} = x$.

Key Property: $\frac{d\, I(x)}{dx} = a(x)\, I(x)$ (Chain Rule)

Assume y is a solution to $y' + a(x)y = b(x)$

$$\Rightarrow \quad \frac{d}{dx}\left(I(x)\,y\right) = I(x)\,y' + I'(x)\,y \qquad \text{(Product Rule)}$$

$$\Rightarrow \quad \frac{d}{dx}\left(I(x)\,y\right) = I(x)\,y' + a(x)\,I(x)\,y$$

$$= I(x)\left(y' + a(x)\,y\right)$$

$$= I(x).\,b(x)$$

$$\Rightarrow \quad I(x)\,y = \int I(x).\,b(x)\,dx$$

general antiderivative

$$\Rightarrow \quad y = \frac{1}{I(x)} \cdot \int I(x).\,b(x)\,dx$$

Conclusion: To find general solution to $y' + a(x)\,y = b(x)$

1/ Find $A(x)$ an antiderivative of $a(x)$

2/ $\boxed{y = \frac{1}{e^{A(x)}}\left(\int e^{A(x)}.\, b(x)\,dx\right)}$ is general solution.

Remarks

1/ You do not need to rederive this formula everytime you solve a Linear equation. Take it as fact.

2/ This formula gets <u>all</u> solutions. We don't need different methods for constant vs. non-constant solutions like in the separable case.

3/ Remember $\int e^{A(x)} \cdot b(x)\,dx$ includes $+C$. That means you will always have $+ \frac{C}{e^{A(x)}}$ in your final answer.

<u>Example</u> $y' + \frac{1}{x} y = \frac{2\sin(x)}{x}$ $(x>0)$ $\Rightarrow$ $a(x) = \frac{1}{x}$, $b(x) = \frac{2\sin(x)}{x}$

$\Rightarrow$ Let $A(x) = \ln(x) \Rightarrow e^{A(x)} = x$

$\Rightarrow y = \frac{1}{x}\left(\int x \cdot \frac{2\sin(x)}{x}\,dx\right)$ a general solution.

$$\Rightarrow \quad y = \frac{1}{x}\int 2\sin(x)\,dx = \frac{1}{x}\left(-2\cos(x) + C\right)$$

$$= \frac{-2\cos(x)}{x} + \frac{C}{x} \quad \text{is a general solution}$$

(C any constant)

<u>Example</u> $\frac{1}{3t^2}y' + y = 4 \quad (t > 0)$

$$\frac{1}{3t^2}y' + y = 4 \Rightarrow y' + 3t^2 y = 12t^2 \Rightarrow \begin{array}{l} a(t) = 3t^2 \\ b(t) = 12t^2 \end{array}$$

Let $A(t) = t^3 \Rightarrow y = \frac{1}{e^{t^3}} \cdot \left(\int e^{t^3} \cdot 12t^2\,dt\right)$

Let $u = t^3 \Rightarrow \frac{du}{dt} = 3t^2 \Rightarrow dt = \frac{du}{3t^2}$

$$\Rightarrow \int e^{t^3} \cdot 12t^2\,dt = \int e^u \cdot 4 \cdot du = 4e^u + C = 4e^{t^3} + C$$

$$\Rightarrow y = \frac{1}{e^{t^3}} \cdot \left(4e^{t^3} + C\right) = 4 + Ce^{-t^3}$$

<u>Remark</u> $\frac{1}{3t^2}y' + y = 4 \Rightarrow \frac{dy}{dt} = 3t^2(4 - y)$

$\Rightarrow$ Could also have used separation of variables.

Here Linear method is better as it captures <u>all</u> solutions. We don't need to find constant / non-constant solutions separately.

<u>Example</u> Solve the following initial value problem :

$$xy' + 2y = x\ln(x) \qquad y(1) = 1. \qquad (x > 0)$$

$xy' + 2y = x\ln(x) \Rightarrow y' + \frac{2}{x} y = \ln(x) \Rightarrow$ $a(x) = \frac{2}{x}$, $b(x) = \ln(x)$

Let $A(x) = 2\ln(x) = \ln(x^2) \Rightarrow e^{A(x)} = x^2$

$$\Rightarrow \quad y = \frac{1}{x^2}\int x^2 \ln(x)\, dx$$

Let $f(x) = \ln(x)$ $\quad g(x) = x^2$

$f'(x) = \frac{1}{x}$ $\quad G(x) = \frac{1}{3}x^3$ $\quad \Rightarrow \int x^2 \ln(x)\, dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2 dx$

$$= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$$

$$\Rightarrow y = \frac{1}{3}x\ln(x) - \frac{1}{9}x + \frac{C}{x^2}$$

$$y(1) = 1 \Rightarrow \frac{-1}{4} + c = 1 \Rightarrow c = \frac{10}{4} \Rightarrow$$

$$y = \frac{1}{3} x \ln(x) - \frac{1}{4} x + \frac{10}{4x^2}.$$