Lagrange Multipliers and Constrained Optimization

Aim : Maximize / Minimize $f(x,y)$ subject to constraint $g(x,y) = 0$

Example Minimize $\overset{f(x,y)}{\overset{\|}{x^2+y^2}}$ subject to constraint $\overset{g(x,y)}{\overset{\|}{x+y-4}} = 0$

Method 1 (From 10A / 16A / 1A) : Solve constraint in y (or x), then substitute into $f(x,y)$, then find max / min with single variable techniques.

$x+y-4 = 0 \Rightarrow y = -x+4$

$\Rightarrow x^2+y^2 = x^2+(-x+4)^2 = h(x)$

$h'(x) = 2x - 2(-x+4) = 0 \Rightarrow 4x - 8 = 0 \Rightarrow x = 2$

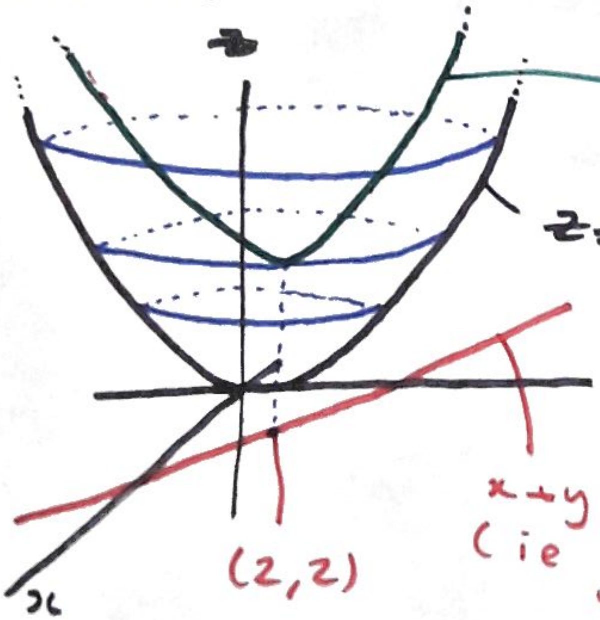
$\underset{2}{\overset{- \quad\quad +}{\longleftarrow\!\!+\!\!\longrightarrow}}$ $h'(x)$ $\Rightarrow$ $h(x)$ has min at $x=2$

$\Rightarrow f(x,y) = x^2+y^2$ has min at $(+2, +2)$ $\left(\begin{array}{c}\text{under constraint} \\ x+y-4=0\end{array}\right)$

Geometric Approach :

We seek the highest / lowest point in the surface $z = f(x,y)$ on the path given by $g(x,y) = 0$

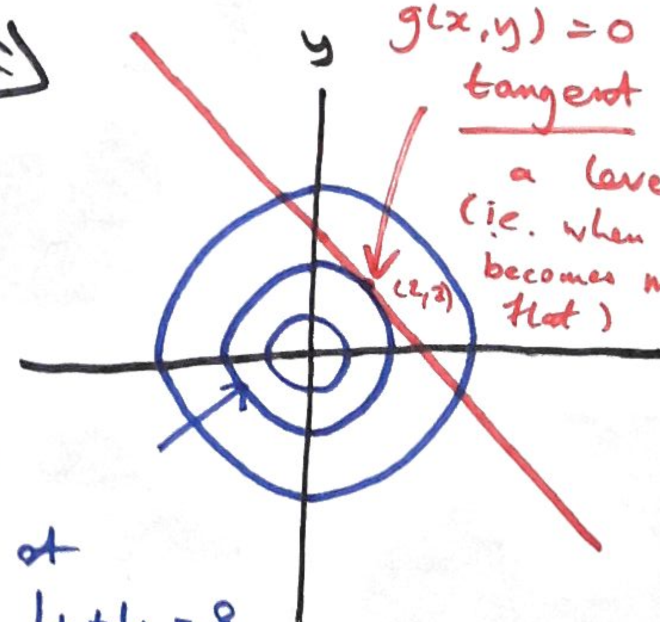
Example



path in $z = f(x,y) = x^2+y^2$ given by $g(x,y) = x+y-4 = 0$

$z = x^2+y^2$. Level curves are circles with centre $(0,0)$

$x+y-4=0$
(ie $y = -x+4$)



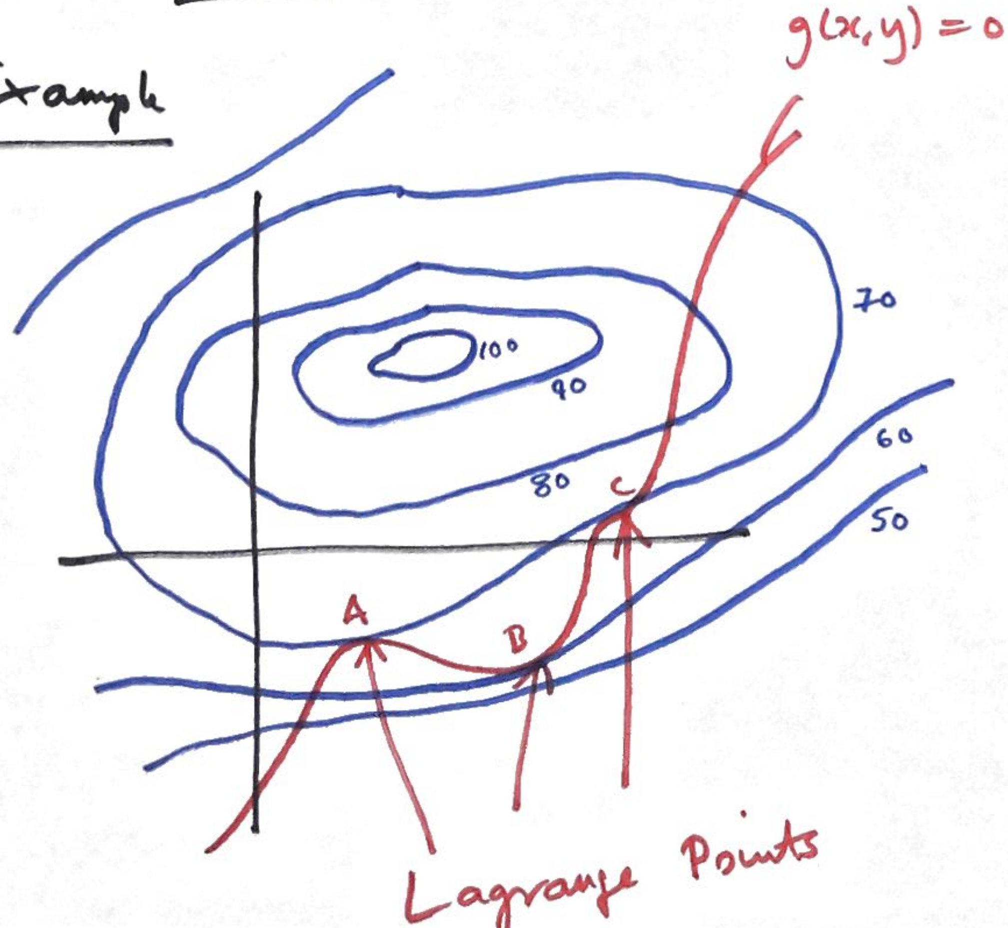
Min is where $g(x,y)=0$ is <u>tangent</u> to a level curve.
(ie. when path becomes momentarily flat)

Level curve at height $\underset{2^2+2^2}{4+4} = 8$

<u>Conclusion</u> : $f(x,y)$ has a max / min at (a,b) under the constraint $g(a,b) = 0$

$\Rightarrow$ At (a,b) the curve $g(x,y) = 0$ is <u>tangent</u> to level curve. We call such a point a <u>Lagrange Point</u>.

<u>Example</u>



A = Relative max

B = Relative min

C = Inflection (neither max nor min)

So Max / Min $\Rightarrow$ Lagrange

<u>but</u>

Lagrange $\not\Rightarrow$ Max / Min

<u>Aim</u> : Find Lagrange Points of $f(x,y)$ subject to $g(x,y) = 0$.

Method (Lagrange Multipliers) :

A/ Define new 3 - variable function :

$$F(x, y, \lambda) = f(x,y) + \lambda g(x,y)$$

λ: new variable

B/ Calculate $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ and $\dfrac{\partial F}{\partial \lambda}$.

C/ Find all solutions to $\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$, $\dfrac{\partial F}{\partial \lambda} = 0$

(a,b,c) a solution $\Rightarrow$ (a,b) is a Lagrange Point.

Hard to see. Not obvious

$\Rightarrow$ (a,b) potential max / min of $f(x,y)$ such that $g(x,y) = 0$.

<u>Remark</u> This gives us <u>potential</u> locations for max / mins.

In this course, we will not worry about verifying if max or min. when thinking about constrained optimization

Example $f(x,y) = x^2+y^2$, $g(x,y) = x+y-4$

A/ $F(x,y,\lambda) = x^2+y^2 + \lambda x + \lambda y - 4\lambda$

always equal $g(x,y)$

B/ $\frac{\partial F}{\partial x} = 2x + \lambda$, $\frac{\partial F}{\partial y} = 2y + \lambda$, $\frac{\partial F}{\partial \lambda} = x+y-4$

c/
$$\begin{aligned} 2x+\lambda &= 0 \\ 2y+\lambda &= 0 \\ x+y-4 &= 0 \end{aligned} \Rightarrow \begin{aligned} \lambda &= -2x \\ \lambda &= -2y \end{aligned} \Rightarrow -2x = -2y \Rightarrow y = x$$

Solve 1st and 2nd in λ

Equate

solve in y (or x)

$$x+y-4=0 \Rightarrow x+x-4=0 \Rightarrow x=2 \Rightarrow y=2 \Rightarrow \lambda=-4$$

Substitute $y=x$

$\Rightarrow$ $(2,2,-4)$ is only solution to $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial \lambda}$.

$\Rightarrow$ $(2,2)$ is only Lagrange Point of x^2+y^2 subject to $x+y-4=0$, as expected.

Example Find potential locations for max/min of

$f(x,y) = 5x^2+6y^2-xy$ subject to $g(x,y) = x+2y-24 = 0$

A/ $F(x,y,\lambda) = 5x^2+6y^2-xy + \lambda x + 2\lambda y - 24\lambda$

B/ $\frac{\partial F}{\partial x} = 10x - y + \lambda$, $\frac{\partial F}{\partial y} = 12y - x + 2\lambda$, $\frac{\partial F}{\partial \lambda} = x+2y-24$

C/ $10x - y + \lambda = 0$ $\Rightarrow$ $\lambda = y - 10x$
$12y - x + 2\lambda = 0$ $\quad$ $\lambda = \frac{x-12y}{2}$ $\Rightarrow$ $y - 10x = \frac{x-12y}{2}$

$\Rightarrow y - 10x = \frac{1}{2}x - 6y \Rightarrow 7y = \frac{21}{2}x \Rightarrow y = \frac{21}{14}x = \frac{3}{2}x$

$x+2y-24 = 0 \Rightarrow x + 2\cdot\frac{3}{2}x - 24 = 0 \Rightarrow 4x - 24 = 0 \Rightarrow x = 6$

$\Rightarrow y = 9 \Rightarrow \lambda = -51$

$\Rightarrow (6,9,-51)$ is only solution

$\Rightarrow (6,9)$ is only potential location for a max/min

of $5x^2+6y^2-xy$ such that $x+2y-24=0$.

Remark

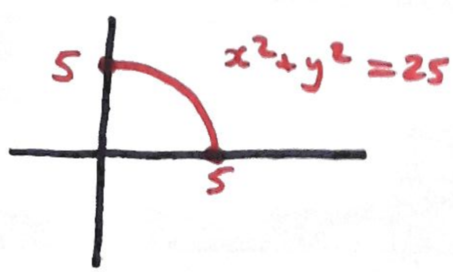
If constraint has endpoints then we can find max / min by first finding Lagrange points, then evaluating $f(x,y)$ at Lagrange points and endpoints.

Example A company makes 2 products A and B. If they make x units of A and y units of B then they are constrained by $x^2 + y^2 = 25$. Suppose they make \$3 dollar profit per unit of A and \$4 dollar profit of B. What are the maximum and minimum possible profits?

$$f(x,y) = 3x + 4y$$

$$g(x,y) = x^2 + y^2 - 25 = 0 \quad \Rightarrow \quad F(x,y,\lambda) = 3x + 4y + \lambda x^2 + \lambda y^2 - 25\lambda$$

$x, y \geq 0$



$$\Rightarrow \frac{\partial F}{\partial x} = 3 + 2\lambda x = 0 \qquad \Rightarrow \qquad \lambda = \frac{-3}{2x} \qquad \Rightarrow \frac{-3}{2x} = \frac{-4}{2y} \Rightarrow 4x = 3y$$

$$\frac{\partial F}{\partial y} = 4 + 2\lambda y = 0 \qquad\qquad \lambda = \frac{-4}{2y}$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 25 = 0$$

$$\Rightarrow \quad y = \frac{4}{3}x$$

$$x^2 + y^2 - 25 = 0 \quad \Rightarrow \quad x^2 + \left(\frac{4}{3}x\right)^2 = 25 \quad \Rightarrow \quad \frac{25}{9}x^2 = 25$$

$$\Rightarrow \quad x^2 = 9 \Rightarrow x = 3 \qquad (x \geq 0)$$

$$\Rightarrow \quad y = 4 \Rightarrow \lambda = \frac{-1}{2}$$

$\Rightarrow \left(3, 4, \frac{-1}{2}\right)$ only solution $\Rightarrow$ $(3,4)$ only Lagrange point

Must check $f(x,y) = 3x + 4y$ at $(3,4)$, $(5,0)$ and $(0,5)$

endpoints

$$f(3,4) = 3^2 + 4^2 = 25$$

$$f(5,0) = 3 \cdot 5 = 15$$

$$f(0,5) = 4 \cdot 5 = 20$$

$\Rightarrow$ \$25 is max at $x = 3$, $y = 4$

\$15 is min at $x = 5$, $y = 0$

The method of Lagrange Multipliers works for 3-variable Functions:

Maximize / Minimize $f(x,y,z)$ subject to $g(x,y,z)=0$.

A/ $F(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$

B/ Calculate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial \lambda}$.

C/ Solve $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0, \frac{\partial F}{\partial \lambda}=0$

(a,b,c,d) solution $\Rightarrow$ (a,b,c) potential max/min of $f(x,y,z)$ such that $g(x,y,z)=0$

Example Find potential max of $f(x,y,z) = xyz$ under constraint

$$\underbrace{xy + xz + yz - 3}_{g(x,y,z)} = 0 \quad (x,y,z \geq 0)$$

A/ $F(x,y,z,\lambda) = xyz + \lambda xy + \lambda xz + \lambda yz - 3\lambda$

B/ $\frac{\partial F}{\partial x} = zy + \lambda y + \lambda z$, $\frac{\partial F}{\partial z} = xy + \lambda x + \lambda y$

$\frac{\partial F}{\partial y} = xz + \lambda x + \lambda z$, $\frac{\partial F}{\partial \lambda} = xy + xz + yz - 3$

always $g(x,y,z)$

$$zy + \lambda y + \lambda z = 0$$
$$xz + \lambda x + \lambda z = 0$$
$$xy + \lambda x + \lambda y = 0$$
$$xy + xz + yz - 3 = 0$$

$\xrightarrow{}$ (solve $1^{st}, 2^{nd}, 3^{rd}$ in λ)

$$\lambda = \frac{-zy}{y+z}$$
$$\lambda = \frac{-xz}{x+z}$$
$$\lambda = \frac{-xy}{x+y}$$

$\Rightarrow$ (Equate)

$$\frac{-zy}{y+z} = \frac{-xz}{x+z}$$
$$\frac{-xz}{x+z} = \frac{-xy}{x+y}$$

$\Rightarrow$ (Simplify)

1/ $\frac{-zy}{y+z} = \frac{-xz}{x+z} \Rightarrow \frac{-y}{y+z} = \frac{-x}{x+z} \Rightarrow \cancel{-xy} - yz = \cancel{-xy} - xz$

$\Rightarrow y = x$

2/ $\frac{-xz}{x+z} = \frac{-xy}{x+y} \Rightarrow \frac{-z}{x+z} = \frac{-y}{x+y} \Rightarrow -xz - \cancel{zy} = -xy - \cancel{zy}$

$\Rightarrow z = y$

$\Rightarrow x = y = z \Rightarrow$

$xy + xz + yz - 3 = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1$
$(x \geq 0)$

$\Rightarrow y = 1 \Rightarrow z = 1 \Rightarrow \lambda = \frac{-1}{2} \Rightarrow (1, 1, 1, \frac{-1}{2})$ only solution

$\Rightarrow (1, 1, 1)$ only potential max.

Secret (Non-Examinable):

If $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$ define

$$D(x,y,\lambda) = \frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$$ (Like in 2-variable case)

If (a,b,c) is a solution to $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial \lambda} = 0$ then

1/ $D(a,b,c) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b,c) > 0 \Rightarrow$ min

2/ $D(a,b,c) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b,c) < 0 \Rightarrow$ Max