<u>Integration by Parts</u> (= Product Rule done in reverse)

Product Rule : $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$

Take antiderivatives of both sides :

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

$$\Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Alternate Form : $G'(x) = g(x)$ ie $G(x)$ an antiderivative of $g(x)$

$$\Rightarrow \boxed{\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx}$$

Integration by parts formula.

When to use :

1/ Integrating a product $f(x)g(x)$ such that can easily find $G(x)$ an antiderivative for $g(x)$

2/ $\int f'(x)G(x)\,dx$ is easier to calculate.

Examples

Q1/ $\int x\cos(x)\,dx = ?$ $\qquad f(x) = x \qquad g(x) = \cos(x)$

$f'(x) = 1 \qquad G(x) = \sin(x)$

derivative $\qquad$ antiderivative

$$\Rightarrow \int x\cos(x)\,dx = x\sin(x) - \int 1\cdot\sin(x)\,dx$$
$$= x\sin(x) - (-\cos(x)) + C$$
$$= x\sin(x) + \cos(x) + C$$

Q2/ $\int x e^{2x}\,dx = ?$ $\qquad f(x) = x \qquad g(x) = e^{2x}$

$f'(x) = 1 \qquad G(x) = \frac{1}{2}e^{2x}$

$$\Rightarrow \int x e^{2x}\,dx = \frac{1}{2}x e^{2x} - \int 1\cdot\frac{1}{2}e^{2x}\,dx = \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + C$$

Q3/ Sometimes it will <u>not</u> be easy to spot $f(x)$ and $g(x)$ which help. E.g. $\int \ln(x)\,dx = ?$

Let $f(x) = \ln(x)$, $g(x) = 1$

$f'(x) = \frac{1}{x}$, $G(x) = x$

$$\Rightarrow \int \ln(x)\,dx = x\ln(x) - \int \frac{1}{x}\cdot x\,dx = x\ln(x) - x + C$$

Q4/ Sometimes it might be necessary to do integration by parts more than once. E.g. $\int x^2 e^x\,dx = ?$

A/ $f(x) = x^2$, $g(x) = e^x$

$f'(x) = 2x$, $G(x) = e^x$

$$\Rightarrow \int x^2 e^x\,dx = x^2 e^x - \int 2x e^x\,dx$$

B/ $f(x) = 2x$ $\quad g(x) = e^x$

$f'(x) = 2$ $\quad G(x) = e^x$

$$\Rightarrow \int 2x e^x\,dx = 2xe^x - \int 2e^x\,dx = 2xe^x - 2e^x + C$$

$$\Rightarrow \int x^2 e^x\,dx = x^2 e^x \underbrace{- 2xe^x + 2e^x}_{-\int 2x e^x\,dx} + C$$

← arbitrary constant

Q5/ $\int e^x \cos(x)\,dx = ?$

A/ $f(x) = \cos(x) \quad g(x) = e^x$

$f'(x) = -\sin(x) \quad G(x) = e^x$

$$\Rightarrow \int e^x \cos(x)\,dx = \cos(x)e^x - \int -e^x \sin(x)\,dx$$

$$= \cos(x)e^x + \int e^x \sin(x)\,dx$$

B/ $f(x) = \sin(x) \quad g(x) = e^x$

$f'(x) = \cos(x) \quad G(x) = e^x$

$$\Rightarrow \int e^x \sin(x)\,dx = \sin(x)e^x - \int e^x \cos(x)\,dx$$

$$\Rightarrow \int e^x \cos(x)\,dx = \cos(x)e^x + \int e^x \sin(x)\,dx$$

$$= \cos(x)e^x + \sin(x)e^x - \int e^x \cos(x)\,dx$$

$$\Rightarrow 2\int e^x \cos(x)\,dx = (\cos(x) + \sin(x))e^x + C$$

$$\Rightarrow \int e^x \cos(x)\,dx = \left(\frac{\cos(x) + \sin(x)}{2}\right) e^x + C$$

Q6/ Sometimes one must use both integration by parts and substitution. Eg. $\int x^5 \cos(x^3)\,dx = ?$

A/ Let $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$

$$\Rightarrow \int x^5 \cos(x^3)\,dx = \int \frac{1}{3} x^3 \cos(x^3)\,du = \frac{1}{3}\int u \cos(u)\,du$$

B/ $f(u) = u \qquad g(u) = \cos(u)$

$f'(u) = 1 \qquad G(u) = \sin(u)$

$$\Rightarrow \int u\cos(u)\,du = u\sin(u) - \int \sin(u)\,du = u\sin(u) + \cos(u) + C$$

$$\Rightarrow \int x^5 \cos(x^3)\,dx = \frac{1}{3} u\sin(u) + \frac{1}{3}\cos(u) + C$$

$$= \frac{1}{3} x^3 \sin(x^3) + \frac{1}{3}\cos(x^3) + C.$$