Infinite Series

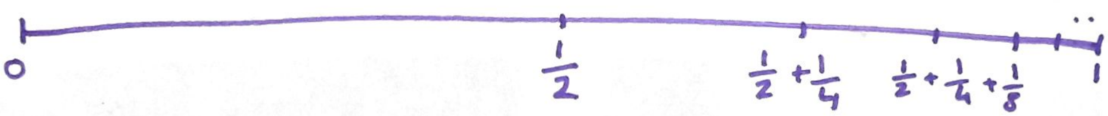
$c_1, c_2, c_3, \ldots$ — sequence of numbers

An <u>infinite series</u> is an "infinite sum" $c_1 + c_2 + c_3 + \ldots$

What does this mean?

<u>Example</u>

$c_1 = \frac{1}{2}, \; c_2 = \frac{1}{4}, \; c_3 = \frac{1}{8}, \; c_4 = \frac{1}{16}, \ldots$



As we add more terms the sum approaches 1. So we might say

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$

<u>Example</u> $c_1 = 1, \; c_2 = 1, \; c_3 = 1, \; c_4 = 1, \ldots$

As we add more terms the sum grows positively without bound.

$\Rightarrow$ $1 + 1 + 1 \ldots$ has no meaning.

<u>Intuitive Definition</u> : 1/ The infinite series $c_1 + c_2 + c_3 + \ldots$ is <u>convergent</u>, with value c, if the finite sums approach c as the number of terms increases.

2/ The infinite series $c_1 + c_2 + c_3 + \ldots$ is <u>divergent</u> if the finite sums do <u>not</u> approach a fixed value.

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ convergent with value 1

$1 + 1 + 1 + \ldots$ divergent

"Sigma notation"

<u>Notation</u> : $c_1 + c_2 + c_3 + \ldots + c_n = \sum_{i=1}^{n} c_i = \sum_{k=1}^{n} c_k$

finite sum

Dummy variable.
Just used to index the terms

$c_1 + c_2 + c_3 + \ldots = \sum_{i=1}^{\infty} c_i$

infinite series

E.g. $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$

$$\sum_{k=1}^{4} \sin\left(\frac{k}{2}\pi\right) = \sin\left(\frac{\pi}{2}\right) + \sin(\pi) + \sin\left(\frac{3\pi}{2}\right) + \sin(2\pi)$$
$$= 1 + 0 + -1 + 0 = 0$$

We can also change where the count starts:

$$\sum_{j=3}^{\infty} c_j = c_3 + c_4 + c_5 + \ldots$$

Important Example : <u>Geometric Series</u>

$a + ar + ar^2 + ar^3 + \ldots$ = geometric series $r \neq 0$

Convergent or Divergent ? (we'll assume $a \neq 0$ otherwise every term is 0 so obviously convergent)

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ = sum of first n terms

<u>Clever trick</u> : $rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$

$$\Rightarrow \quad S_n - rS_n = a + \cancel{ar} + \cancel{ar^2} + \ldots + \cancel{ar^{n-1}} - \cancel{ar} - \cancel{ar^2} - \ldots - \cancel{ar^{n-1}} - ar^n$$

$$= a - ar^n$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r}$$

$|r| < 1 \Rightarrow r^n \to 0$ as $n \to \infty$

$|r| > 1 \Rightarrow r^n$ grows without bound as n grows

Conclusions :

$$a + ar + ar^2 + \ldots = \begin{cases} \frac{a}{1-r} \text{ (ie convergent) if } |r|<1 \\ \text{Divergent if } |r| > 1 \end{cases}$$

$|r| = 1 \Rightarrow r = \pm 1 \Rightarrow$ $\left.\begin{matrix} a + a + a \ldots \\ \text{or} \\ a - a + a - a \ldots \end{matrix}\right\}$ Both divergent

Why do we care ?

Example (Drug Dosage)

On the first of each month a patient is to be given 10 mg of a drug. Over any month long period, the amount of the drug in the patient's system will drop by 20%. In the long term, how much of the drug will be in their system <u>just after</u> a new dose is given ?

Beginning of Month 1 : 10 mg

Beginning of Month 2 : $0.8 \times 10 + 10 = 10 + 10\cdot(0.8)$ mg

80% of 10   new dose

" " " 3 : $0.8 \times (0.8 \times 10 + 10) + 10 = 10 + 10\cdot(0.8) + 10(0.8)^2$ mg

80% of previous month   new dose

" " " 4 : $10 + 10\cdot(0.8) + 10\cdot(0.8)^2 + 10\cdot(0.8)^3$ mg

$\Rightarrow$ In long term the patient will have

$10 + 10\cdot(0.8) + 10\cdot(0.8)^2 + \dots$ mg after a dose is given

$10 + 10\cdot(0.8) + 10\cdot(0.8)^2 + \ldots$ = geometric series with $a = 10$, $r = 0.8$

$\Rightarrow$ Patient will have $\frac{10}{1-0.8} = 50$ mg in their system after each dose in long term.

Example (Multiplier Effect in Economics)

The government enacts a tax cut of \$10 billion. Assume that each person will spend 90% (called marginal propensity to consume) of all resulting extra income and save the rest. How much total extra spending will result.

1^{st} spending : 90% of \$10 billion $= 0.9 \times 10$ billion dollars

2^{nd} spending : 90% of 0.9×10 billion dollars $= (0.9)^2 \times 10$ billion dollars

3^{rd} spending : 90% of $(0.9)^2 \times 10$ billion dollars $= (0.9)^3 \times 10$ billion dollar

$\vdots$

geometric series with $a = 10\cdot(0.9) = 9$, $r = 0.9$

$\Rightarrow$ Total Extra Spending $= 10(0.9) + 10\cdot(0.9) + 10\cdot(0.9)^2 + \ldots$ billion dollars

$= \frac{9}{1-0.9} =$ \$ 90 billion dollars