Infinite Series with Positive Terms

Q/ : Someone makes you the following offers :

1/ You will be paid \$1000 today, followed by \$500 tomorrow, followed by \$250 the next day, etc

2/ You will be paid \$1 today, followed by \$$\frac{1}{2}$ tomorrow, followed by \$$\frac{1}{3}$ the next day, etc.

Which offer do you take?

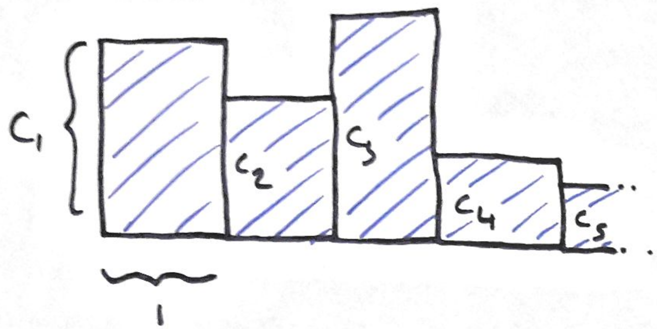
Total from 1/ $= 1000 + 500 + 250 + \dots$ ← geometric series $a = 1000$, $r = \frac{1}{2}$

$$= \frac{1000}{1 - \frac{1}{2}} = \$2000$$

Total from 2/ $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = ?$ ← not a geometric series.

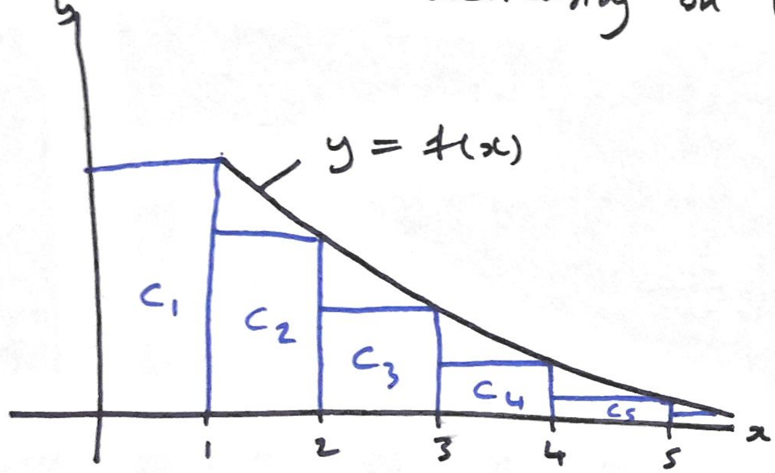
$c_1, c_2, c_3, \dots$ − sequence of positive numbers.

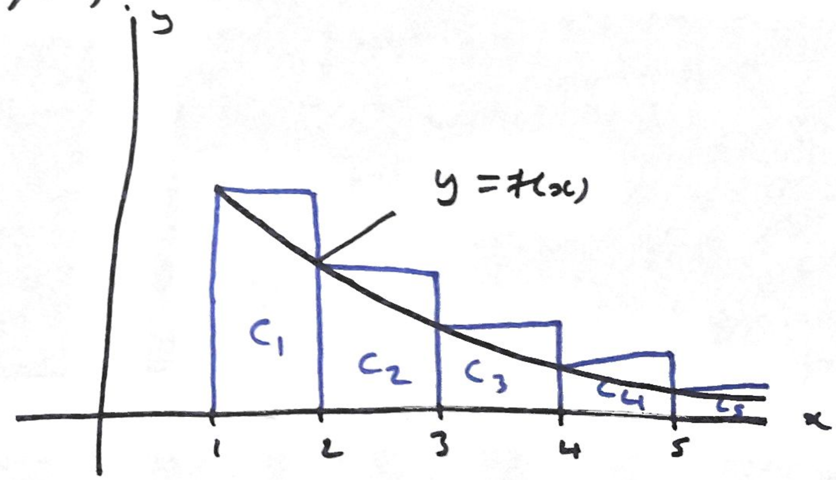
Need a test to determine if $c_1 + c_2 + c_3 + \dots$ convergent or divergent.



$c_1 + c_2 + c_3 + \dots$ convergent $\iff$ Area finite

Let $f(x)$ be a function such that $f(n) = c_n$ for all n.
Assume $f(x)$ is decreasing on $[1, \infty)$.





$$\Rightarrow c_2 + c_3 + c_4 + \dots \leq \int_1^\infty f(x)\,dx \leq c_1 + c_2 + c_3 + \dots$$

Area under $y = f(x)$ over $[1, \infty)$

Hence $c_1 + c_2 + c_3 \dots$ convergent $\Rightarrow \int_1^\infty f(x)\,dx$ convergent <u>and</u>

$\int_1^\infty f(x)\,dx$ convergent $\Rightarrow c_2 + c_3 + \dots$ convergent $\Rightarrow c_1 + c_2 + \dots$ convergent

The Integral Test

Let $c_1, c_2, c_3, \dots$ be a sequence of positive numbers. Let $f(x)$ be a function such that

1/ $f(n) = c_n$ for all n positive whole numbers

2/ $f(x)$ is <u>decreasing</u> on $[1, \infty)$

then $c_1 + c_2 + c_3 + \dots$ convergent $\Leftrightarrow \int_1^\infty f(x)\,dx$ convergent.

<u>Example</u> $1 + \frac{1}{2} + \frac{1}{3} + \dots = ?$

$f(x) = 1/x \Rightarrow$ 1/ $f(n) = \frac{1}{n}$ for all positive whole numbers.

2/ $f(x)$ decreasing on $[1, \infty)$

$$\int_1^t \frac{1}{x}\,dx = \ln|x|\Big|_1^t = \ln|t| \Rightarrow$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|t| = \infty \quad (\text{Divergent})$$

$\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \dots$ is divergent by integral test.

<u>Answer to first question</u> : Take after 2/. The sum grows positively without bound.

<u>Example</u> Is $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ convergent or divergent ?

$f(x) = \frac{1}{x^2} \Rightarrow$ 1/ $f(n) = \frac{1}{n^2}$ for all whole numbers

2/ $f'(x) = \frac{-2}{x^3} < 0$ for $x \geq 1 \Rightarrow f(x)$ decreasing on $[1, \infty)$

$$\int_1^t \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^t = \frac{-1}{t} - \frac{-1}{1} = 1 - \frac{1}{t}$$

$$\Rightarrow \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = 1 \Rightarrow \text{convergent}$$

$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ __convergent__.

__Warning__ : $\int_1^\infty \frac{1}{x^2}\,dx = 1 \quad \not\Rightarrow \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = 1$

The test only tells us if convergent or divergent. It does __not__ give us the value if convergent.

Using $f(x) = \frac{1}{x^p}$ (p any number) we can show the following.

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots = \begin{cases} \text{Convergent if} & p > 1 \\ \text{Divergent if} & p \leq 1 \end{cases}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^p} \longleftarrow \text{called a } p\text{-series}$$

__Remark__ We can slightly weaken the conditions of the integral test. For example we only need $f(x)$ to be decreasing past a certain point, ie on $[N, \infty)$ for some $N \geq 1$.

<u>Example</u> $\sum_{n=1}^{\infty} \frac{\ln(n)}{n} = \frac{\ln(1)}{1} + \frac{\ln(2)}{2} + \dots$ convergent or divergent ?

$f(x) = \frac{\ln(x)}{x} \Rightarrow$ 1/ $f(n) = \frac{\ln(n)}{n}$ for all positive whole numbers

2/ $f'(x) = \frac{\frac{1}{x}x - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}$

$f'(x) < 0 \Leftrightarrow 1 - \ln(x) < 0 \Leftrightarrow x > e$

$\Rightarrow f(x)$ decreasing on $[e, \infty) \Rightarrow$ Can apply integral test.

Let $u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du \Rightarrow$

$$\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(x))^2 + C$$

$$\Rightarrow \int_1^t \frac{\ln(x)}{x}\,dx = \frac{1}{2}(\ln(x))^2 \Big|_1^t = \frac{1}{2}(\ln(t))^2$$

still use 1

$$\Rightarrow \int_1^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \frac{1}{2}(\ln(t))^2 = \infty \quad \text{(Divergent)}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ divergent.

Comparison Test

Take two infinite series $a_1 + a_2 + a_3 + \dots$ and $b_1 + b_2 + b_3 + \dots$

Assume $0 \leq a_n \leq b_n$ for all n.

$\Rightarrow$ 1/ $a_1 + a_2 + a_3 + \dots$ __divergent__ $\Rightarrow$ $b_1 + b_2 + b_3 + \dots$ __divergent__

2/ $b_1 + b_2 + b_3 + \dots$ __convergent__ $\Rightarrow$ $a_1 + a_2 + a_3 + \dots$ __convergent__.

__Examples__ 1/ $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ divergent (p series with $p = 1$)

$$\Rightarrow \sum_{n=1}^{\infty} \frac{e^n}{n} = \frac{e}{1} + \frac{e^2}{2} + \frac{e^3}{3} + \dots \text{ divergent}$$

because $0 \leq \frac{1}{n} \leq \frac{e^n}{n}$ for all $n \geq 1$.

2/ $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ __conv__ or __div__ ?

Recall $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ convergent.

Also $0 \leq |\sin(n)| \leq 1$ for all n $\Rightarrow$ $0 \leq \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$ for all $n \geq 1$.

Hence $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ convergent.

3/ $\sum_{k=1}^{\infty} \frac{3}{1+4^k}$ <u>conv</u> or <u>div</u> ?

Not geometric series or p-series. Let's compare to $\sum_{k=1}^{\infty} \frac{3}{4^k}$

$= \frac{3}{4} + \frac{3}{4^2} + \dots$ = geometric series with $r = \frac{1}{4}$ $\Rightarrow$ convergent

$0 \leq \frac{3}{1+4^k} \leq \frac{3}{4^k}$ for all k $\Rightarrow$ $\sum_{k=1}^{\infty} \frac{3}{1+4^k}$ <u>conv</u>.

Basic Advice: Is there a reasonably obvious geometric or p-series to do a comparison with?