Improper Integrals

Reminder on Limits

$$\lim_{t \to \infty/-\infty} f(t) = L \iff$$ $f(t)$ tends to L as t grows positively / negatively without bound

If there is no such L we say the limit does not exist (DNE)

Example

$$\lim_{t \to \infty/-\infty} f(t) = \infty/-\infty \iff$$ $f(t)$ grows positively / negatively without bound as t grows positively / negatively without bound.

Core Examples :

$$\lim_{t \to \infty} e^{kt} = \begin{cases} \infty & \text{if } k > 0 \\ 0 & \text{if } k < 0 \end{cases}, \quad \lim_{t \to -\infty} e^{kt} = \begin{cases} 0 & \text{if } k > 0 \\ \infty & \text{if } k < 0 \end{cases}$$

$$\lim_{t \to \infty} t^p = \begin{cases} \infty & \text{if } p > 0 \\ 0 & \text{if } p < 0 \end{cases}, \quad \lim_{t \to \infty} \ln(t) = \infty$$

$$\lim_{t\to\infty} \frac{p(t)}{q(t)} = \begin{cases} 0 & \text{if } \deg(p(t)) < \deg(q(t)) \\ \pm\infty & \text{if } \deg(p(t)) > \deg(q(t)) \\ \text{ratio of leading coefficients} & \text{if } \deg(p(t)) = \deg(q(t)) \end{cases}$$

Warning: Limits are not about plugging ∞ into functions:

∞ is not a number. $f(\infty)$ is meaningless. Limits are about understanding the output of the function as the input grows without bound.

Stupid example: $\frac{2}{\infty} = 0 = 1/\infty \Rightarrow 2 = 1$

Meaningless

Definition

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$$

variable

Improper Integrals

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^{b} f(x)\,dx$$

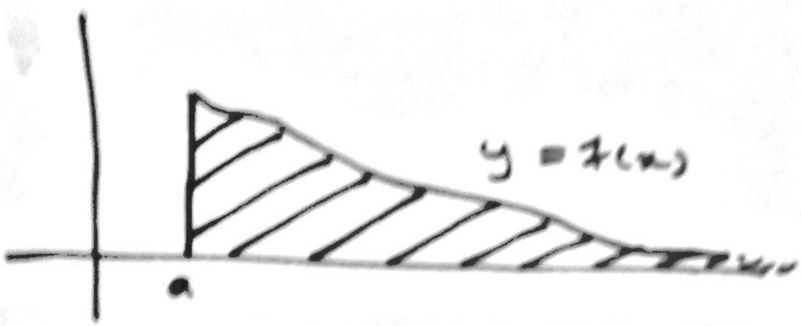
variable

If these limits exist we say the improper integral is <u>convergent</u>. If not we say it is <u>divergent</u>.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} f(x)\,dx + \int_{-\infty}^{0} f(x)\,dx$$

convergent if and only if both convergent.

Geometric Interpretation:



$$\int_{a}^{\infty} f(x)\,dx = \text{Area}(\text{▨})$$

Convergent $\iff$ Area finite

<u>Examples</u> 1) $\displaystyle\int_{0}^{\infty} e^{-x}\,dx = ?$

$$\int e^{-x}\,dx = -e^{-x} + C \implies \int_{0}^{t} e^{-x}\,dx = -e^{-x}\Big|_{0}^{t} = 1 - e^{-t}$$

$$\Rightarrow \int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} 1 - e^{-t} = 1 - 0 = 1 \quad (\underline{\text{convergent}})$$

2/ $$\int_{-\infty}^{-1} \frac{1}{x}\,dx = ?$$

$$\int \frac{1}{x}\,dx = \ln|x| + c \Rightarrow \int_t^{-1} \frac{1}{x}\,dx = \ln|x|\Big|_t^{-1}$$

$$= \ln|-1| - \ln|t| = -\ln|t|$$

$$\Rightarrow \int_{-\infty}^{-1} \frac{1}{x}\,dx = \lim_{t\to-\infty} -\ln|t| = -\infty \Rightarrow \underline{\text{Divergent}}$$

3/ (Capital Value)

Capital Value at continuous income stream = Present value at continuous income stream over $[0,\infty)$ = $\displaystyle\int_0^\infty f(t)\,e^{-rt}\,dt$

interest rate

income rate

Find the capital value of an asset that generates income at a rate of \$5000 per year assuming an interest rate of 10%.

$f(t) = 5000$

$r = 0.1$ $\Rightarrow$ Capital Value $= \int_0^\infty 5000\, e^{-0.1t}\, dt$

$$\int_0^b 5000\, e^{-0.1t}\, dt = -50000\, e^{-0.1t} \Big|_0^b = 50000 - 50000\, e^{-0.1b}$$

$\Rightarrow$ Capital value $= \lim_{b \to \infty} 50000 - 50000\, e^{-0.1b} = 50000$

$\left(\lim_{b \to \infty} e^{-0.1b} = 0 \right)$

Hence the capital value is \$50000.

4/ $\int_0^\infty \frac{e^{-x}}{(e^{-x}+1)^2}\, dx = ?$

(We'll need to do substitution here. Always calculate indefinite integral before going back to improper integral.)

• Let $u = e^{-x}+1 \Rightarrow \frac{du}{dx} = -e^{-x} \Rightarrow dx = \frac{du}{-e^{-x}}$

$$\Rightarrow \int \frac{e^{-x}}{(e^{-x}+1)^2}\,dx = -\int \frac{1}{(e^{-x}+1)^2}\,du = -\int \frac{1}{u^2}\,du = \frac{1}{u} + C$$

$$= \frac{1}{e^{-x}+1} + C$$

$$\Rightarrow \int_0^t \frac{e^{-x}}{(e^{-x}+1)^2}\,dx = \frac{1}{e^{-x}+1}\Bigg|_0^t = \frac{1}{e^{-t}+1} - \frac{1}{2}$$

$$\Rightarrow \int_0^\infty \frac{e^{-x}}{(e^{-x}+2)^2}\,dx = \lim_{t\to\infty}\left(\frac{1}{e^{-t}+1} - \frac{1}{2}\right) = \frac{1}{0+1} - \frac{1}{2} = \frac{1}{2}$$

$$\left(\lim_{t\to\infty} e^{-t} = 0\right)$$