Graphing Solutions of Differential Equations

$$\frac{dy}{dx} = q(y)$$ – **Autonomous** 1^{st}-Order Diff. Eq.

(Autonomous $\Rightarrow$ Separable)

Strategy to find all solutions :

1/ Solve $q(y) = 0$ giving all constant solutions.

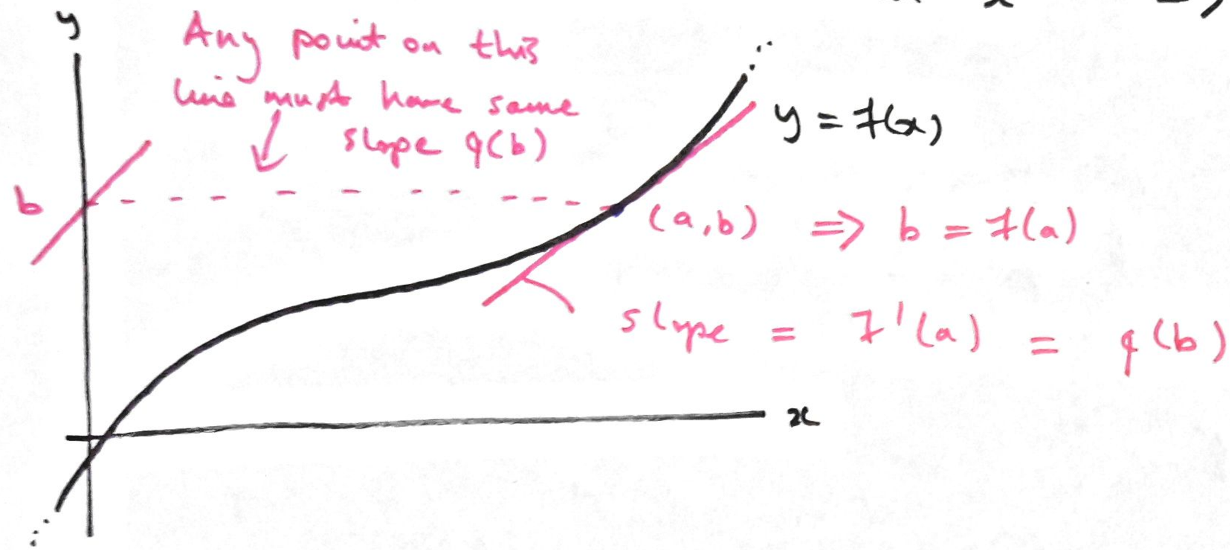
2/ $\int \frac{1}{q(y)}\, dy = \int 1 . dx$ gives all non-constant solutions.

Potential Problems : Perhaps $q(y)$ is not given explicitly. For example, as a table or graph. Even if we have $q(y)$ explicitly it may still be impossible to calculate $\int \frac{1}{q(y)}\, dy$. For example if $q(y) = y^2 + y - 2$ we cannot calculate $\int \frac{1}{y^2+y-2}\, dy$.

Aim : Find a way to sketch (roughly) possible solutions to $\frac{dy}{dx} = q(y)$ without explicitly solving it.

Assume $y = f(x)$ a solution to $\frac{dy}{dx} = g(y)$.

$\Rightarrow$ $f'(x) = g(f(x))$ for all x $\Rightarrow$



Conclusion : Slope at $(a,b) = g(b)$

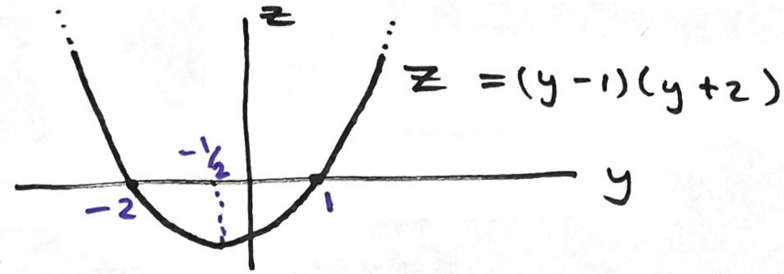
Strategy 1/ On the y-axis sketch line segments with slope $g(b)$ for multiple b.

2/ Given initial condition $y(0) = y_0$ start at $(0, y_0)$ and draw a curve whose slopes match the slopes at these line segments. This will give a crude sketch of the solution.

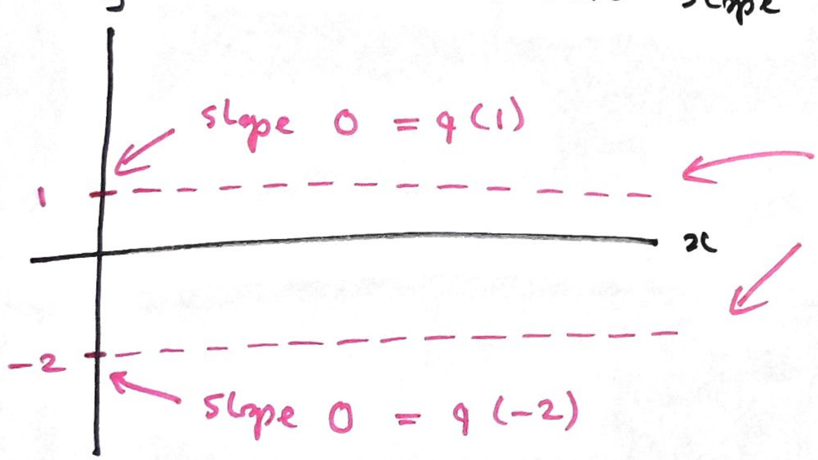
Example $\dfrac{dy}{dx} = y^2 + y - 2$ $\quad y(0) = 1$, $\; y(0) = \frac{1}{2}$, $\; y(0) = -3$

1/ $q(y) = y^2 + y - 2 = (y-1)(y+2)$

Sketch $z = q(y)$:

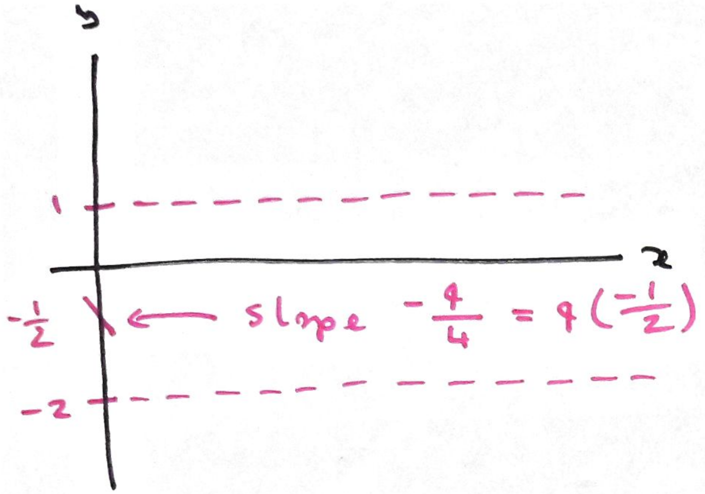


a) First mark the zero slope segments. In this case $q(1) = q(-2) = 0$

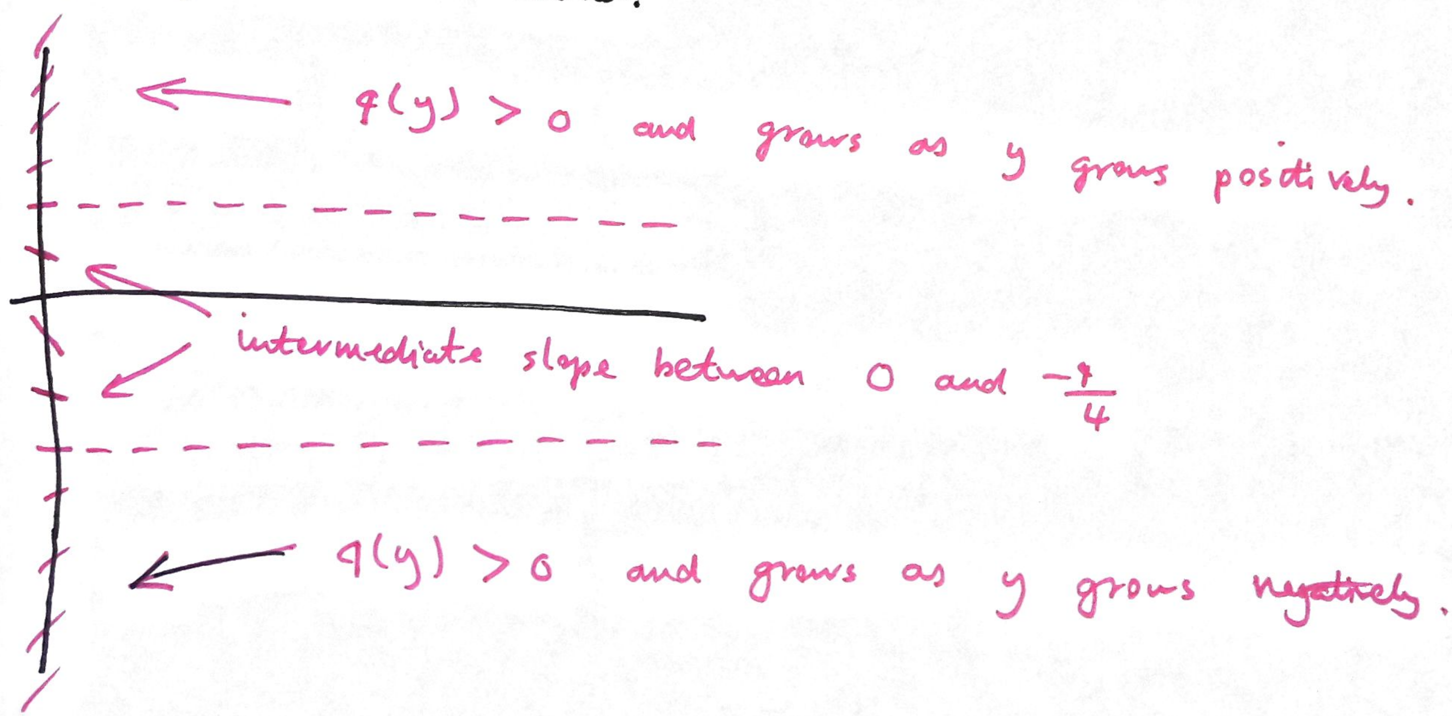


These will be useful guidelines

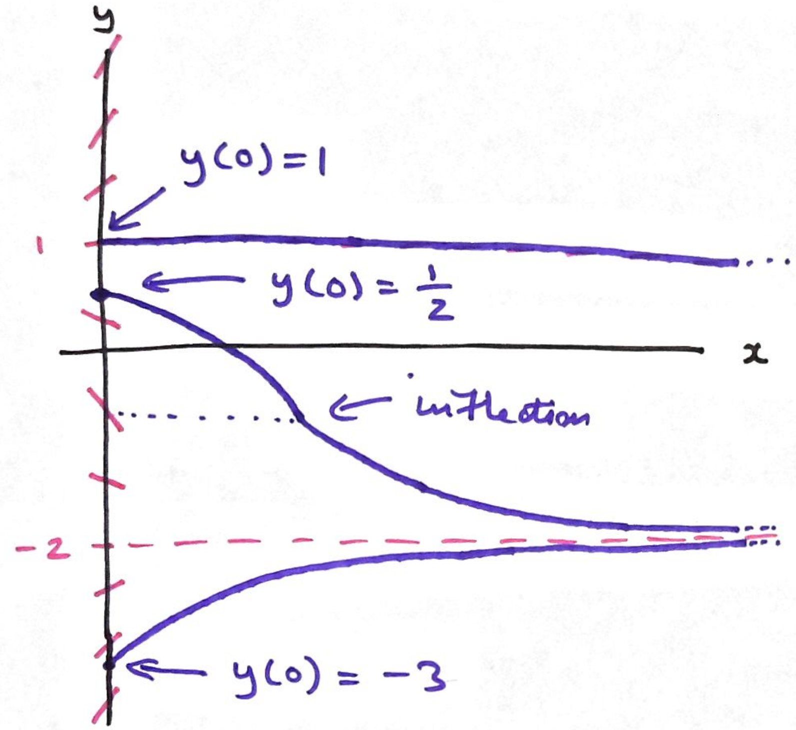
b). Next mark max/min slopes. In this case min at $y = -\frac{1}{2}$.
(You don't need to do this exactly). $q(\frac{-1}{2}) = -\frac{9}{4}$



c) Now draw in intermediate line segments. Also draw in line segments reflecting the behavior of $q(y)$ as y grows positively / negatively without bound.



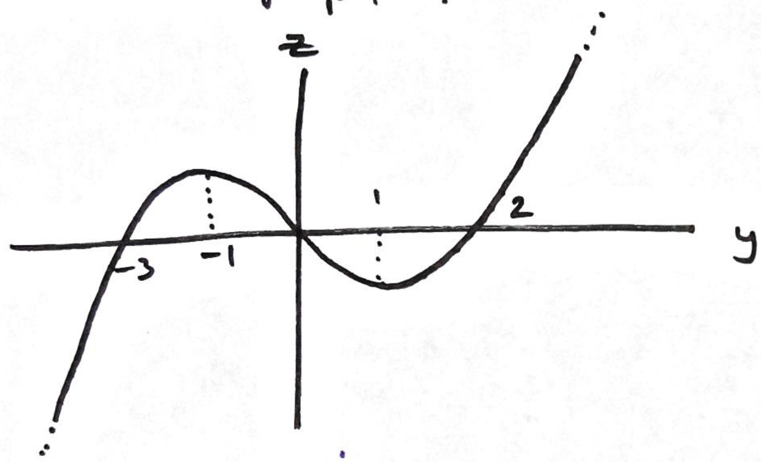
2/ We must now sketch solutions each with a different starting position : $(0,1)$, $(0,\frac{1}{2})$ and $(0,-3)$



Remarks 1/ Even though we don't have formulae for these solutions we still know a lot. For example if $y(0)=\frac{1}{2}$ and $\frac{dy}{dx}=y^2+y-2$ then $y \to -2$ as $x \to \infty$.

2/ The hardest part about this technique is correctly sketching the line segments on the y-axis. The best way to get good is practice.

<u>Example</u> Sketch solutions to $\frac{dy}{dx} = q(y)$ with initial conditions $y(0) = -2$, $y(0) = \frac{1}{2}$, $y(0) = 3$, $y(0) = -4$ where $q(y)$ has graph :



$\Rightarrow$ z-coordinate will be the slope of the line segment we draw on the y-axis.

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