<u>Review</u>

$f(x,y), g(x,y)$ - 2 variable functions

<u>Aim</u> : Maximize / Minimize $f(x,y)$ subject to constraint $g(x,y) = 0$

<u>Method</u> : Lagrange Multipliers

1/ Define $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

new variable

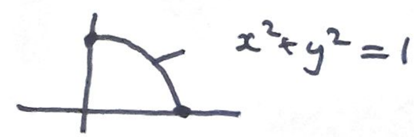
2/ Write 3 equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial \lambda} = 0$

3/ Solve first 2 in λ and equate. Solve this in one variable (x or y).

4/ Substitute into $\frac{\partial F}{\partial \lambda} = 0$ (ie $g(x,y) = 0$). Solve in remaining variable.

5/ This gives triples (a,b,c). The points (a,b) are the potential locations of maxima/minima.

Lagrange points

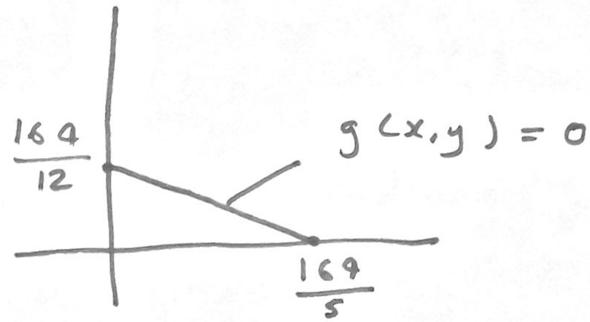


6/ If constraint has endpoints, e.g. $x^2+y^2-1=0$ and $x,y \geq 0$ evaluate at endpoints <u>and</u> Lagrange points to find absolute max/min

<u>Example</u> Find <u>max</u> value of $f(x,y) = x^2+y^2$ subject to constraint

$12y + 5x = 169$ where $x, y \geq 0$.

$g(x,y) = 12y + 5x - 169$



$F(x,y,\lambda) = x^2+y^2 + 12\lambda y + 5\lambda x - 169\lambda$

$$\Rightarrow \left.\begin{aligned} \frac{\partial F}{\partial x} &= 2x + 5\lambda = 0 \\ \frac{\partial F}{\partial y} &= 2y + 12\lambda = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda &= \frac{-2x}{5} \\ \lambda &= \frac{-2y}{12} \end{aligned} \Rightarrow \frac{-2x}{5} = \frac{-2y}{12} \Rightarrow x = \frac{5}{12}y$$

$$\frac{\partial F}{\partial \lambda} = 12y + 5x - 169 = 0 \Rightarrow 12y + 5\cdot\left(\frac{5}{12}y\right) = 169 \Rightarrow \frac{12^2+5^2}{12}y = 169 \Rightarrow y = 12$$

$\Rightarrow x = 5 \Rightarrow \lambda = -2$

$\Rightarrow (5,12)$ is only Lagrange Point

$F(5,12) = 169$

$F(\frac{169}{5}, 0) = 169 \cdot \frac{169}{25} \leftarrow$ max value

$F(0, \frac{169}{12}) = 169 \cdot \frac{169}{144}$

Can do same with 3-variable functions. No endpoint problems in this case.

<u>Example</u> Find <u>max</u> value of xyz such that $36 - x - 6y - 3z = 0$

(Assume a max exists)

$$F(x,y,z,\lambda) = xyz + 36\lambda - \lambda x - 6\lambda y - 3\lambda z \Rightarrow$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= yz - \lambda = 0 \\ \frac{\partial F}{\partial y} &= xz - 6\lambda = 0 \\ \frac{\partial F}{\partial z} &= xy - 3\lambda = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \lambda &= zy \\ \lambda &= \frac{xz}{6} \\ \lambda &= \frac{xy}{3} \end{aligned}\right\} \Rightarrow \begin{aligned} zy &= \frac{xz}{6} \\ zy &= \frac{xy}{3} \end{aligned} \Rightarrow \begin{aligned} y &= \frac{x}{6} \\ z &= \frac{x}{3} \end{aligned} \Rightarrow \begin{aligned} x &= 6y \\ x &= 3z \end{aligned}$$

equate to get 2 new equations

Solve 2 variables in one unknown

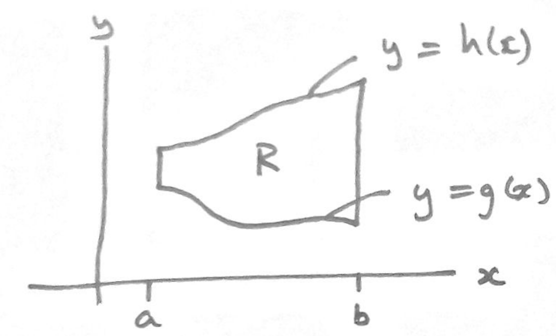
$$\frac{\partial F}{\partial \lambda} = 36 - x - 6y - 3z = 0 \Rightarrow 36 - x - x - x = 0 \Rightarrow x = 12$$

$\Rightarrow y = 2 \Rightarrow z = 4 \Rightarrow \lambda = 8$

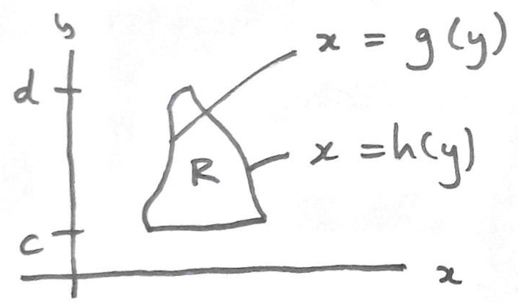
$\Rightarrow (12, 2, 4)$ is only Lagrange point

$\Rightarrow 12 \cdot 2 \cdot 4 = 96$ is <u>max</u> value.

Double Integrals



$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$



$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$

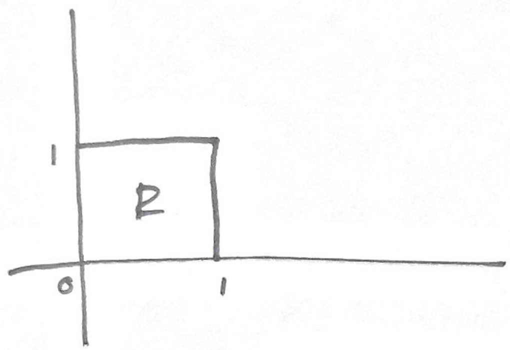
Sometimes we'll need to break R into pieces of each type and sum up. Double integrals are used to construct average value of 2-variable $f(x,y)$ on R.

Average value of $f(x,y)$ on R $= \dfrac{1}{\text{Area}(R)} \displaystyle\iint_R f(x,y)\,dx\,dy$

Practical example : Average cost of producing 2 products.

Examples

1/ R region with $0 \le x \le 1$, $0 \le y \le 1$. Calculate $\iint_R 2x^3 e^{x^2 y}\,dx\,dy$.



$\Rightarrow$ <u>2 approaches</u> $\int_0^1 \left(\int_0^1 2x^3 e^{x^2 y}\,dx \right) dy$ or

$\int_0^1 \left(\int_0^1 2x^3 e^{x^2 y}\,dy \right) dx$

y thought of as constant

$\int_0^1 2x^3 e^{x^2 y}\,dx$ is very hard. Choose 2^{nd} method.

x thought of as constant

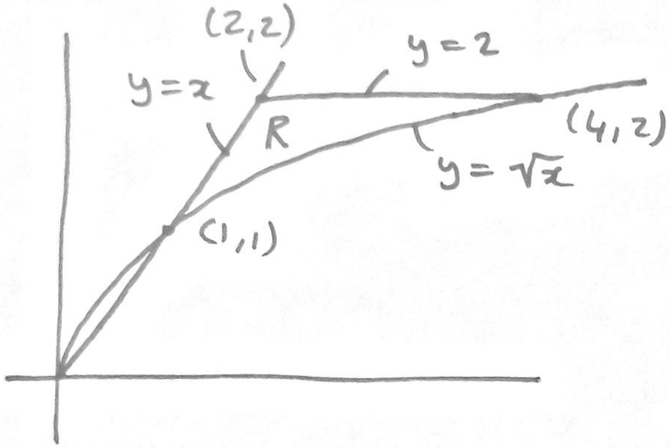
$$\int_0^1 2x^3 e^{x^2 y}\,dy = 2x e^{x^2 y} \Big|_0^1 = 2x e^{x^2} - 2x$$

$$\Rightarrow \iint_R 2x^3 e^{x^2 y}\,dx\,dy = \int_0^1 2x e^{x^2} - 2x\,dx = e^{x^2} - x^2 \Big|_0^1 = e - 1 - 1 = e - 2.$$

2/ Let R be the region bounded by $y = \sqrt{x}$, $y = x$ and $y = 2$

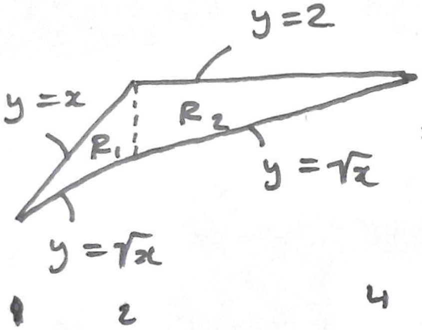
Calculate $\iint_R xy\,dx\,dy$.

$\sqrt{x} = x \Rightarrow x = 1 \Rightarrow y = x$ and $y = \sqrt{x}$ cross at $(1,1)$



Again there are 2 approaches :

1/

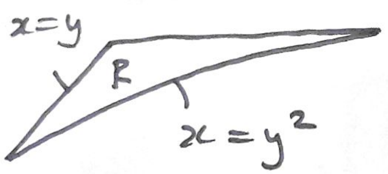


$\Rightarrow \iint_R xy\,dx\,dy = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy$

$$= \int_1^2 \left(\int_{\sqrt{x}}^{x} xy\,dy \right) dx + \int_2^4 \left(\int_{\sqrt{x}}^{2} xy\,dy \right) dx$$

don't get this wrong!

2/



$\Rightarrow \iint_R xy\,dx\,dy = \int_1^2 \left(\int_y^{y^2} xy\,dx \right) dy$

$$= \int_1^2 \frac{1}{2}x^2y \Big|_y^{y^2} dy = \int_1^2 \frac{1}{2}y^5 - \frac{1}{2}y^3\,dy = \frac{1}{12}y^6 - \frac{1}{8}y^4 \Big|_1^2$$

$$= \frac{64}{12} - \frac{16}{8} - \frac{1}{12} + \frac{1}{8} = \frac{27}{8}$$

Both are fine. The 2nd is less work.

Vital Integration Techniques :

Integration by Substitution : $\int f(g(x))\, g'(x)\, dx = ?$

$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)} \Rightarrow$

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

antiderivative of f

Integration by Parts : $\int f(x)\, g(x)\, dx = ?$

Hopefully easier

$$G'(x) = g(x) \Rightarrow \int f(x)\, g(x)\, dx = f(x)\, G(x) - \int f'(x)\, G(x)\, dx$$

Pro advice : When calculating definite integral, work out indefinite integral first.

Improper Integrals : $\int_a^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx$, $\int_{-\infty}^{b} f(x)\, dx = \lim_{t \to -\infty} \int_t^b f(x)\, dx$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} f(x)\,dx + \int_{-\infty}^{0} f(x)\,dx$$ ← must be calculated separately.

Convergent $\iff$ all limit exist.

Make sure you can differentiate / integrate all core functions, e^x, $\ln(x)$, x^n, $\sin(x)$, $\cos(x)$, ...

Important Applications : income rate

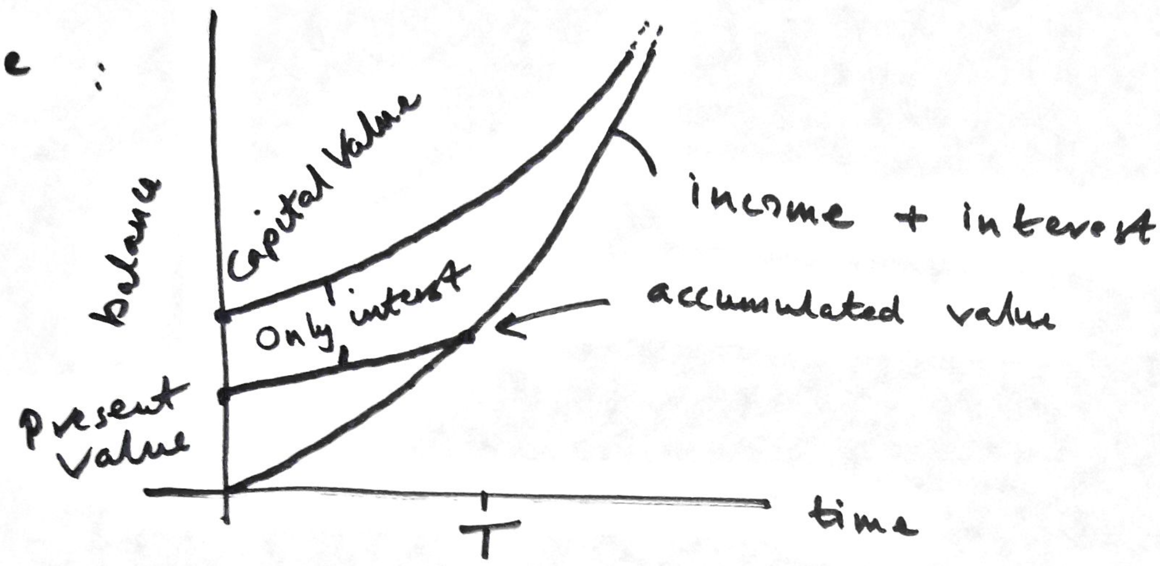
interest rate

Accumulated Value over $[0,T]$ $= \displaystyle\int_0^T f(t)\, e^{r(T-t)}\,dt$ = Amount in account at T with both income and interest being added

Present Value over $[0,T]$ $= \displaystyle\int_0^T f(t)\, e^{-rt}\,dt$ = Amount needed at time 0 to have accumulated value at T adding only interest.

Capital Value $= \displaystyle\int_0^\infty f(t)\, e^{-rt}\,dt$ = Amount needed at time 0 such that balance approaches true balance adding only interest.

Picture :



<u>Example</u> A company has <u>constant</u> income rate. They invest their earnings in an account with 10% interest rate. If the company has capital value \$1000000 what is their income rate?

$f(t) = k$ (constant)

$$\Rightarrow 1000000 = \int_0^\infty k e^{-\frac{1}{10}t}\, dt$$

$$\int k e^{-\frac{1}{10}t}\,dt = -10k e^{-\frac{1}{10}t} + C$$

$$\Rightarrow \int_0^b k e^{-\frac{1}{10}t}\,dt = -10k e^{-\frac{1}{10}t}\Big|_0^b = 10k - 10k e^{-\frac{1}{10}b}$$

$$\Rightarrow \int_0^\infty k e^{-\frac{1}{10}t}\,dt = \lim_{b \to \infty} 10k - 10k e^{-\frac{1}{10}b} = 10k$$

$\Rightarrow \quad 10k = 1000000 \quad \Rightarrow k = \$100{,}000$ per year.

Differential Equations

$$\frac{dy}{dx} = p(x)\,q(y) \quad - \text{ Separable}$$

<u>Aim</u> : Find general solution

1/ $q(y) = 0$ gives all <u>constant</u> solutions e.g. $y(x) = 1$ for all x.

remember + C

2/ $\int \frac{1}{q(y)}\,dy = \int p(x)\,dx$ gives all <u>non-constant</u> solutions

$\frac{dy}{dx} + a(x)y = b(x)$ - Linear

remember $+C$ in bracket

General solution (both constant and non-constant) $= \frac{1}{e^{A(x)}}\left(\int e^{A(x)} b(x)\,dx\right)$

$(A'(x) = a(x))$

Practical Examples : Population growth + emmigration/immigration ; Savings account + withdrawals / deposits ; Loan + payments

$\frac{dy}{dx} = q(y)$ - autonomous

If cannot calculate $\int \frac{1}{q(y)}\,dy$ use graphic methods.

Tough Example : $\sqrt{x}\,y' = y\tan(\sqrt{x})$

Method 1 Separable : $y' = y \cdot \frac{\tan(\sqrt{x})}{\sqrt{x}}$ $\Rightarrow$ $y=0$ only constant solution.

Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du \Rightarrow$

$$\int \frac{\tan(\sqrt{x})}{\sqrt{x}}\,dx = \int 2\tan(u)\,du = \int \frac{2\sin(u)}{\cos(u)}\,du$$

$$v = \cos(u) \Rightarrow \frac{dv}{du} = -\sin(u) \Rightarrow du = \frac{dv}{-\sin(u)}$$

$$\Rightarrow \int 2\tan(u)\,du = \int \frac{-2}{v}\,dv = -2\ln|v| + C$$

$$= -2\ln|\cos(u)| + C = -2\ln|\cos(\sqrt{x})| + C$$

$$= \ln|\cos^{-2}(\sqrt{x})| + C = \ln(\sec^2(\sqrt{x})) + C$$

$$\int \frac{1}{y}\,dy = \int \frac{\tan(\sqrt{x})}{\sqrt{x}}\,dx \Rightarrow \ln|y| = \ln(\sec^2(\sqrt{x})) + C$$

$$\Rightarrow y = \pm e^{C} \cdot e^{\ln(\sec^2(\sqrt{x}))} = \pm e^{C} \sec^2(\sqrt{x})$$

$$\Rightarrow y(x) = \begin{cases} \pm e^{C} \cdot \sec^2(\sqrt{x}) & C \text{ any constant} \\ 0 & \end{cases}$$

is general solution

Method 2 : Linear

$$\sqrt{x}\,y' = y\tan(\sqrt{x}) \Rightarrow y' + \overset{a(x)}{\left(\frac{-\tan(\sqrt{x})}{\sqrt{x}}\right)} y = \overset{b(x)}{0}$$

Previous work

$$\int \frac{-\tan(\sqrt{x})}{\sqrt{x}}\,dx = -\ln(\sec^2(\sqrt{x})) + C$$

$$\Rightarrow \text{ Let } A(x) = -\ln(\sec^2(\sqrt{x})) = \ln(\cos^2(\sqrt{x}))$$

$$\Rightarrow e^{A(x)} = \cos^2(\sqrt{x}) \qquad b(x) = 0$$

$$\Rightarrow y(x) = \frac{1}{\cos^2(\sqrt{x})}\left(\int 0\,dx\right) = \frac{C}{\cos^2(\sqrt{x})}$$

general solution.

Sequences and Series

$a_1, a_2, a_3, \dots$ – sequence of numbers

$s_n = a_1 + a_2 + a_3 + \dots + a_n$ – sum of first n terms

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots = \lim_{n\to\infty} s_n$$

Definition

Limit exists → convergent

Limit DNE → divergent

Core examples:

p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots = \begin{cases} \text{conv.} & \text{if } p > 1 \\ \text{div} & \text{if } p \leq 1 \end{cases}$$

geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots = \begin{cases} \text{conv. with value } \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{div. if } |r| \geq 1 \end{cases}$$

How to test for convergence:

Is it p-series or geometric series? $\xrightarrow{\text{No}}$ If we replace k with x to get $f(x)$ can we calculate $\int f(x)dx$? $\xrightarrow{\text{No}}$ Does it resemble a p-series or geometric series $\xrightarrow{\text{No}}$ Not going to show up!

Is it p-series or geometric series? ↓ Yes → Check numbers

Can we calculate $\int f(x)dx$? ↓ Yes → Integral Test

Does it resemble a p-series or geometric series ↓ Yes → Comparison test

Example $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+7}$, $f(x) = \dfrac{1}{x^2+7}$, $\dfrac{1}{k^2+7}$ looks like $\dfrac{1}{k^2}$

means not a p-series

can't integrate

$0 < \dfrac{1}{k^2+7} < \dfrac{1}{k^2}$ for all k

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ convergent as $p = 2 > 1$ $\overset{\text{C.T.}}{\Longrightarrow}$ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+7}$ convergent.

Integral Test : $\displaystyle\sum_{k=1}^{\infty} c_k$ conv. or div.? Let $f(x)$ be function on $[1, \infty)$ such that

1/ $f(k) = c_k$ for $k \geq 1$ (k integer) (usually automatic as you've chosen $f(x)$ using the c_k)

2/ $f'(x) < 0$ on $[1, \infty)$ $\Rightarrow$ $f(x)$ decreasing on $[1, \infty)$

Must always be checked

then $\displaystyle\int_1^{\infty} f(x)\,dx$ conv. $\Rightarrow$ $\displaystyle\sum_{k=1}^{\infty} c_k$ conv.

and

$\displaystyle\int_1^{\infty} f(x)\,dx$ div. $\Rightarrow$ $\displaystyle\sum_{k=1}^{\infty} c_k$ div.

Taylor Series

Core examples :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

Don't stress about Σ - notation
Just write enough terms to indicate pattern.

Fact : We can manipulate them like polynomials

Example

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\Rightarrow \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \dots$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 \dots$$

$$\Rightarrow \frac{-1}{(1+x)^2} = \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\left(1 - x + x^2 - x^3 + x^4 - x^5 \dots\right)$$

$$= -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \dots$$

<u>Your job</u>: Find a route from core example to desired function.

<u>Example</u> What is Taylor series of xe^{x^2} ?

$$e^{x} \xrightarrow[\text{replace } x \text{ with } x^2]{} e^{x^2} \xrightarrow[\text{multiply by } x \text{ term by term}]{} xe^{x^2}$$

<u>Fact</u> $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ Taylor series of $f(x)$ $\Rightarrow$ $a_n = \frac{f^{(n)}(0)}{n!}$

$\Rightarrow f^{(n)}(0) = a_n \cdot n!$

a_n: coefficient in front of x^n (could be 0)

<u>Example</u>: If $1 - x + x^7 + x^{11} + \dots$ is Taylor series of $f(x)$

$\Rightarrow \frac{f^{(7)}(0)}{7!} = 1 =$ coefficient in front of x^7 $\Rightarrow f^{(7)}(0) = 7!$

$\frac{f^{(6)}(0)}{6!} = 0 =$ coefficient in front of of x^6 $\Rightarrow f^{(6)}(0) = 0 \cdot 6! = 0$

Probability

X - C.R.V. on interval $I = [A, B], [A, \infty), (-\infty, B], (-\infty, \infty)$

Cumulative Distribution Function

$F(x) = Pr(X \leq x)$ $(x \text{ in } I)$

Definition

$Pr(a \leq X \leq b) = F(b) - F(a)$

$F(\text{left endpoint of } I) = 0$

Probability Density Function

$f(x)$ on I such that

1/ $f(x) \geq 0$ on I

2/ $\int_I f(x)\,dx = 1$

Any function on I doing these is a P.D.F. on I.

integral on interval I (e.g. $\int_A^B, \int_A^\infty, \int_{-\infty}^B, \int_{-\infty}^\infty$)

Differentiate ie $F'(x) = f(x)$

Integrate + $F(A) = 0$

$Pr(a \leq X \leq b) = \int_a^b f(x)\,dx$

$E(X) = \int_I x f(x)\,dx$

$Var(X) = \int_I x^2 f(x)\,dx - (E(X))^2$

$Pr(X \leq Median(X)) = \frac{1}{2}$

X - D.R.V. with values $a_1, a_2, a_3, \ldots$

$\Rightarrow$ There are numbers $Pr(X = a_k)$ such that

1/ $Pr(X = a_k) \geq 0$ for all $k \geq 1$

2/ $Pr(X = a_1) + Pr(X = a_2) + Pr(X = a_3) + \cdots = 1$

Important Examples (You need to know these well)

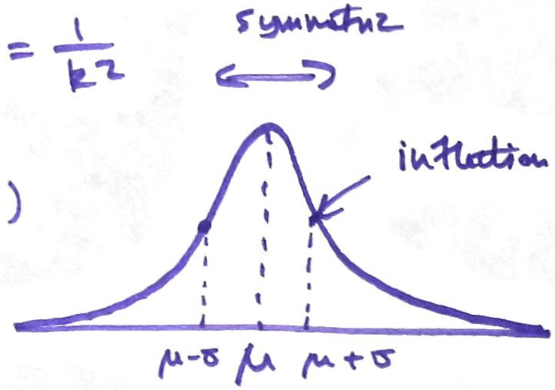
Exponential : X C.R.V. on $[0,\infty)$

$f(x) = ke^{-kx}$ for some $k>0$

$\mathbb{E}(X) = \frac{1}{k}$, $Var(X) = \frac{1}{k^2}$

Normal : X C.R.V. on $(-\infty,\infty)$

$y=f(x)$ a "bell curve"



$\mathbb{E}(X) = Median(X) = \mu$

$Var(X) = \sigma^2$

Standard normal $\mu=0$, $\sigma=1$

$Pr(X\le b) = Pr(Z \le \frac{b-\mu}{\sigma})$

IMPORTANT !!!!

Poisson : X D.R.V. on $0,1,2,3,4\ldots$

$Pr(X=n) = \frac{\lambda^n}{n!}e^{-\lambda}$ $(\lambda>0)$ $\mathbb{E}(X) = \lambda$, $Var(X) = \lambda$

Geometric : X D.R.V. on $0,1,2,3,4\ldots$

$Pr(X=n) = p^n(1-p)$ $(0\le p<1)$, $\mathbb{E}(X) = \frac{p}{1-p}$, $Var(X) = \frac{p}{(1-p)^2}$

Remarks

1/ For Poisson / Geometric $\Pr(X \text{ has certain property})$

$= $ Sum of all $\Pr(X=n)$ where n has that property.

E.g. $\Pr(X \text{ divisible by } 3) = \Pr(X=0) + \Pr(X=3) + \Pr(X=6) + \ldots$

2/ Look at the practical examples in class / homework.