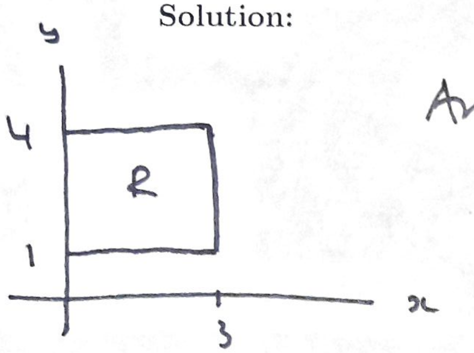
This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) A company makes two products A and B. The cost of making x units of A and y units of B is

$$C(x, y) = xe^x y^2.$$

If the company makes between 0 and 3 units of A and 1 and 4 units of B what are their average costs?

Solution:



$$\text{Area}(R) = 3^2 = 9$$

$$\text{Average Cost} = \frac{1}{9}\iint_R xe^x y^2 \, dx\, dy = \frac{1}{9}\int_1^4\left(\int_0^3 xe^x y^2 \, dx\right) dy$$

$$\int xe^x y^2 \, dx = y^2 \int \underset{\substack{\uparrow \\ f(x)}}{x}\,\underset{\substack{\uparrow \\ g(x)}}{e^x} \, dx = y^2 (xe^x - e^x) + C$$

$f(x)$ $g(x)$, $f'(x) = 1$, $G(x) = e^x$

$$\Rightarrow \int_0^3 xe^x y^2 \, dx = y^2(xe^x - e^x)\Big|_0^3 = y^2(3e^3 - e^3 + 1)$$

$$\Rightarrow \text{Average cost} = \frac{1}{9}\int_1^4 y^2(3e^3 - e^3 + 1)\, dy$$

$$= \frac{1}{9}\cdot\frac{1}{3}y^3(3e^3 - e^3 + 1)\Big|_1^4 = \frac{15}{27}(3e^3 - e^3 + 1).$$

2. (25 points) Find the points (x, y, z) which maximise the function $xy+3xz+3yz$, subject to the constraint $xyz = 9$. You may assume, without justification, that a maximum exists.

Solution:

$$F(x,y,z,\lambda) = xy + 3xz + 3yz + \lambda xyz - 9\lambda$$

$$\left.\begin{aligned}
\frac{\partial F}{\partial x} &= y + 3z + \lambda yz = 0 \\
\frac{\partial F}{\partial y} &= x + 3z + \lambda xz = 0 \\
\frac{\partial F}{\partial z} &= 3x + 3y + \lambda xy = 0
\end{aligned}\right\}
\begin{aligned}
\lambda &= \frac{-(y+3z)}{yz} = \frac{-1}{z} + \frac{-3}{y} \\
\lambda &= \frac{-(x+3z)}{xz} = \frac{-1}{z} + \frac{-3}{x} \\
\lambda &= \frac{-(3x+3y)}{xy} = \frac{-3}{y} + \frac{-3}{x}
\end{aligned}$$

$\Rightarrow \frac{-1}{z} + \frac{-3}{y} = \frac{-1}{z} + \frac{-3}{x}$ and $\frac{-1}{z} + \frac{-3}{y} = \frac{-3}{y} + \frac{-3}{x}$

$\Rightarrow x = y$ and $x = 3z \Rightarrow z = \frac{x}{3}$

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xyz - 9 = 0 \Rightarrow \frac{x^3}{3} = 9 \Rightarrow x = 3$

$\Rightarrow y = 3 \Rightarrow z = 1 \Rightarrow \lambda = -2$

$\Rightarrow (3,3,1)$ is only Lagrange Point $\Rightarrow (3,3,1)$ maximizes function.

3. (25 points) Find the general solution to the following differential equation:

$$2xy' = 6x\sin(x^{3/2}) - y.$$

Solution:

$$2xy' = 6x\sin(x^{3/2}) - y \Rightarrow y' + \underbrace{\frac{1}{2x}}_{a(x)} y = \underbrace{3\sin(x^{3/2})}_{b(x)}$$

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\ln x + C = \ln x^{\frac{1}{2}} + C$$

Let $A(x) = \ln x^{\frac{1}{2}} \Rightarrow e^{A(x)} = x^{\frac{1}{2}}$

$$y(x) = \frac{1}{x^{\frac{1}{2}}} \cdot \int x^{\frac{1}{2}} \cdot 3\sin(x^{3/2})\,dx$$

Let $u = x^{3/2} \Rightarrow \frac{du}{dx} = \frac{3}{2}x^{\frac{1}{2}} \Rightarrow dx = \frac{du}{\frac{3}{2}x^{\frac{1}{2}}} \Rightarrow$

$$\int x^{\frac{1}{2}} \cdot 3 \cdot \sin(x^{3/2})\,dx = \int 2\sin(u)\,du = -2\cos(u) + C$$
$$= -2\cos(x^{\frac{3}{2}}) + C$$

$$\Rightarrow \quad y(x) = \frac{1}{x^{\frac{1}{2}}}\left(-2\cos(x^{3/2}) + C\right)$$

$$= \frac{-2\cos(x^{3/2})}{\sqrt{x}} + \frac{C}{\sqrt{x}}$$

4. (25 points) A company projects that over the time frame $[0, t]$ they will make $10000t^2$ dollars. If they intend to invest their income in an account with a 20% interest rate, what is the present value of the company's earnings over the next year?

Solution:

Income rate $= \frac{d}{dt}(10000t^2) = 20000t$

$r = \frac{1}{5}$

$\Rightarrow$ Present Value over $[0,1]$ $= \int_0^1 20000te^{-\frac{1}{5}t}\,dt$

$$\int \underset{f(t)}{t}\,\underset{g(t)}{e^{-\frac{1}{5}t}}\,dt = -5te^{-\frac{1}{5}t} + 5\int e^{-\frac{1}{5}t}\,dt$$

$f'(t) = 1$ $\quad G(t) = -5e^{-\frac{1}{5}t}$

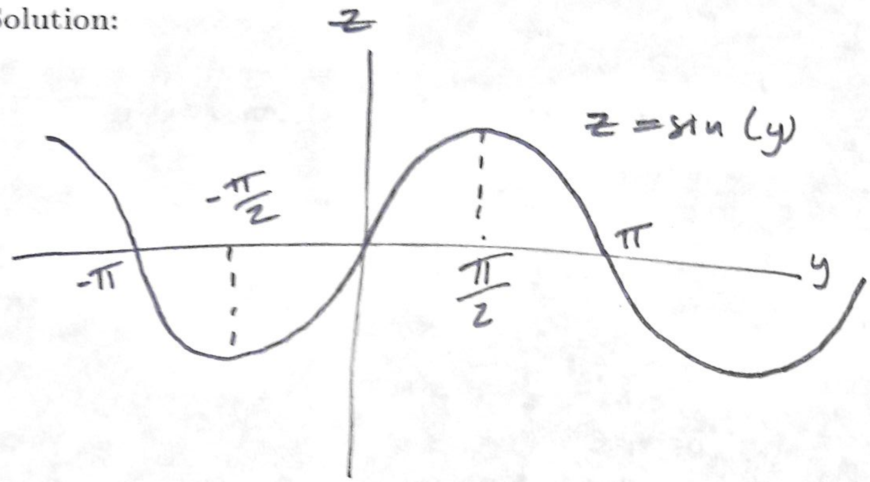
$$= -5te^{-\frac{1}{5}t} - 25e^{-\frac{1}{5}t} + C$$

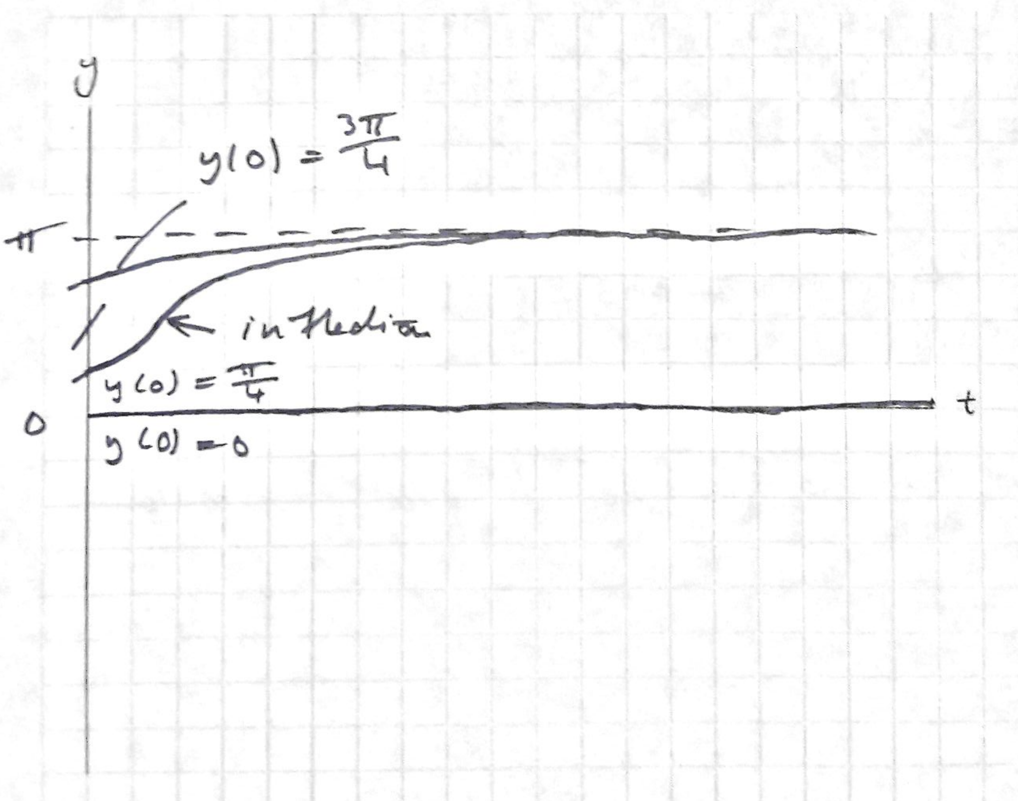
$\Rightarrow$ Present Value over $[0,1]$ $= 20000 \cdot \left(-5te^{-\frac{1}{5}t} - 25e^{-\frac{1}{5}t}\right)\Big|_0^1$

$= 20000\left(-5e^{-\frac{1}{5}} - 25e^{-\frac{1}{5}t} + 25\right)$ dollars.

5. (25 points) Consider the differential equation $y' = \sin(y)$. Graph the solutions with the following initial conditions: $y(0) = 0$, $y(0) = \pi/4$ and $y(0) = 3\pi/4$. Be sure to label each solution carefully.

Solution:





6. (a) (15 points) Using the integral test, determine whether

$$\sum_{k=1}^{\infty} \frac{k}{k^2+1}$$

is convergent or divergent. Be sure to check all the hypotheses of the integral test.

Solution:

$$f(x) = \frac{x}{x^2+1} \Rightarrow$$

1/ $f(k) = \dfrac{k}{k^2+1}$ for $k \geq 1$ integer

2/ $f'(x) = \dfrac{(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$

on $[1, \infty)$

$\Rightarrow f(x)$ decreasing on $[1, \infty)$

$\Rightarrow$ Can apply integral test.

$$u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C$$

$$\Rightarrow \int_1^t \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|\Big|_1^t = \frac{1}{2}\ln(t^2+1) - \frac{1}{2}\ln(2)$$

$$\Rightarrow \int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\ln(t^2+1) - \frac{1}{2}\ln(2) = \infty \Rightarrow \text{Div.}$$

$\Rightarrow \sum_{k=1}^{\infty} \frac{k}{k^2+1}$ divergent.

(b) (10 points) Using this, or otherwise, determine if the infinite series

$$\sum_{k=1}^{\infty} \frac{2k+10}{k^2+1}$$

is convergent or divergent.

Solution:

$$2k+10 > k > 0 \quad \text{for } k \geq 1$$

$$\Rightarrow \frac{2k+10}{k^2+1} > \frac{k}{k^2+1} > 0 \quad \text{for } k \geq 1$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{2k+10}{k^2+1} \text{ divergent by comparison test.}$$

7. (25 points) Determine the Taylor series of the following functions. Write at least the first five non-zero terms of the sum to illustrate the general pattern.

(a)

$$f(x) = \frac{1}{1+x^2}$$

Solution:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

(b)

$$f(x) = x\sin(x^2)$$

Solution:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots$$

$$\Rightarrow \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} \cdots$$

$$\Rightarrow x\sin(x^2) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \frac{x^{19}}{9!} \cdots$$

8. (25 points) Let X be a continuous random variable on $[1, 5]$ with cumulative distribution function $F(x) = \frac{1}{2}\sqrt{x-1}$. Calculate the expected value of X. You do not need to simplify your answer.

$$\text{P.D.F} = \frac{1}{4} \cdot \frac{1}{\sqrt{x-1}} \Rightarrow \mathbb{E}(X) = \int_1^5 x \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{x-1}}\, dx$$

Let $u = x - 1 \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du \Rightarrow$
$(x = u + 1)$

$$\frac{1}{4}\int x \cdot \frac{1}{\sqrt{x-1}}\, dx = \frac{1}{4}\int (u+1) \cdot \frac{1}{\sqrt{u}}\, du$$

$$= \frac{1}{4}\int \sqrt{u} + \frac{1}{\sqrt{u}}\, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} + \frac{1}{4} \cdot 2 \cdot \sqrt{u} + C$$

$$= \frac{1}{6}(x-1)^{3/2} + \frac{1}{2}\sqrt{(x-1)} + C$$

$$\Rightarrow \mathbb{E}(X) = \frac{1}{6}(x-1)^{3/2} + \frac{1}{2}\sqrt{x-1}\,\Big|_1^5$$

$$= \frac{1}{6} 4^{3/2} + \frac{1}{2}\sqrt{4}$$

9. (25 points) Are there any values of k for which

$$f(x) = ke^{-x} + \frac{10}{(x+2)^2}$$

is a probability density function on $[0, \infty)$? Carefully justify your answer. Hint: Consider $f(0)$.

Solution:

$$\int_0^\infty ke^{-x} + \frac{10}{(x+2)^2}\,dx = \lim_{t\to\infty}\left(-ke^{-x} - \frac{10}{(x+2)}\Big|_0^t\right)$$

$$= \lim_{t\to\infty} -ke^{-t} - \frac{10}{t+2} + k + 5 = k+5$$

$f(x)$ P.D.F. $\Rightarrow$ $k+5 = 1$ $\Rightarrow$ $k = -4$

However $-4e^{-x} + \frac{10}{(x+2)^2}$ has value $-4 + \frac{10}{4} < 0$

at $x = 0$ $\Rightarrow$ $f(x)$ is <u>not</u> non-negative

on $[0, \infty)$ $\Rightarrow$ $f(x)$ is <u>not</u> a P.D.F. on $[0, \infty)$

So no value of k gives a P.D.F.

10. (25 points) Let X be a discrete random variable which can take value any non-negative integer $0, 1, 2, 3, \ldots\ldots$ Assume that

$$Pr(X = n) = \frac{1}{2^{n+1}},$$

for any n, a non-negative integer. Calculate the probability that X is greater than or equal to 5 and odd. What is the variance of X?

Solution:

$Pr(X = n) = \left(\frac{1}{2}\right)^n \cdot \frac{1}{2}$ ← geometric with $p = \frac{1}{2}$

$Pr(X \geq 5 \text{ and odd}) = Pr(X=5) + Pr(X=7) + \ldots$

$= \left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^9 \cdot \left(\frac{1}{2}\right) + \ldots$

↑ geometric series with $a = \left(\frac{1}{2}\right)^5 \cdot \frac{1}{2}$ $r = \left(\frac{1}{2}\right)^2$

$\Rightarrow Pr(X \geq 5 \text{ and odd}) = \dfrac{\left(\frac{1}{2}\right)^5 \cdot \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2}$

$Var(X) = \dfrac{p}{(1-p)^2} = \dfrac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = 2.$

END OF EXAM