DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Formulae

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) A company makes two products A and B. The cost of making x units of A and y units of B is
$$C(x, y) = xe^x y^2.$$
If the company makes between 0 and 3 units of A and 1 and 4 units of B what are their average costs?

Solution:

2. (25 points) Find the points (x, y, z) which maximise the function $xy+3xz+3yz$, subject to the constraint $xyz = 9$. You may assume, without justification, that a maximum exists.

   Solution:

3. (25 points) Find the general solution to the following differential equation:

$$2xy' = 6x\sin(x^{3/2}) - y.$$

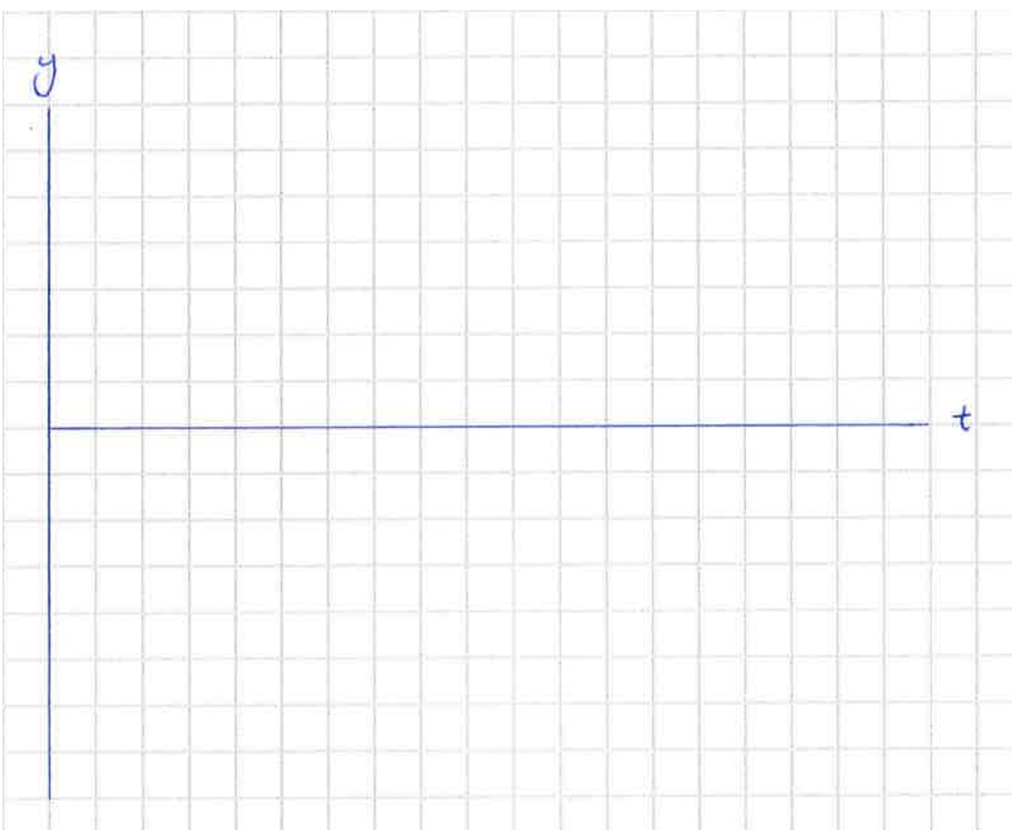
Solution:

4. (25 points) A company projects that over the time frame $[0, t]$ they will make $10000t^2$ dollars. If they intend to invest their income in an account with a 20% interest rate, what is the present value of the company's earnings over the next year?

   Solution:

5. (25 points) Consider the differential equation $y' = \sin(y)$. Graph the solutions with the following initial conditions: $y(0) = 0$, $y(0) = \pi/4$ and $y(0) = 3\pi/4$. Be sure to label each solution carefully.

Solution:



6. (a) (15 points) Using the integral test, determine whether

$$\sum_{k=1}^{\infty} \frac{k}{k^2+1}$$

is convergent or divergent. Be sure to check all the hypotheses of the integral test.

Solution:

(b) (10 points) Using this, or otherwise, determine if the infinite series

$$\sum_{k=1}^{\infty} \frac{2k+10}{k^2+1}$$

is convergent or divergent.

Solution:

7. (25 points) Determine the Taylor series of the following functions. Write at least the first five non-zero terms of the sum to illustrate the general pattern.

   (a)
$$f(x) = \frac{1}{1+x^2}$$

   Solution:

   (b)
$$f(x) = x\sin(x^2)$$

   Solution:

8. (25 points) Let X be a continuous random variable on $[1, 5]$ with cumulative distribution function $F(x) = \frac{1}{2}\sqrt{x-1}$. Calculate the expected value of X. You do not need to simplify your answer.

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9. (25 points) Are there any values of k for which

$$f(x) = ke^{-x} + \frac{10}{(x+2)^2}$$

is a probability density function on $[0, \infty)$? Carefully justify your answer. Hint: Consider $f(0)$.

Solution:

10. (25 points) Let X be a discrete random variable which can take value any non-negative integer $0, 1, 2, 3, ...$ Assume that

$$Pr(X = n) = \frac{1}{2^{n+1}},$$

for any n, a non-negative integer. Calculate the probability that X is greater than or equal to 5 and odd. What is the variance of X?

Solution:

END OF EXAM