This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) Find the points (x, y) which minimize the function $\sqrt{6x^2 + y^2}$, subject to the constraint $2x + y = 5$. You may assume, without justification that a minimum exists.

Solution:

$$F(x,y,\lambda) = \sqrt{6x^2+y^2} + 2\lambda x + \lambda y - 5\lambda$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= \frac{12x \cdot \frac{1}{2}}{\sqrt{6x^2+y^2}} + 2\lambda = 0 \\ \frac{\partial F}{\partial y} &= \frac{2y \cdot \frac{1}{2}}{\sqrt{6x^2+y^2}} + \lambda = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda &= \frac{-3x}{\sqrt{6x^2+y^2}} \\ \lambda &= \frac{-y}{\sqrt{6x^2+y^2}} \end{aligned}$$

$$\Rightarrow -3x = -y \Rightarrow y = 3x$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow 2x + y - 5 = 0 \Rightarrow 2x + 3x - 5 = 0 \Rightarrow x = 1$$

$$\Rightarrow y = 3 \Rightarrow \lambda = \frac{-3}{\sqrt{15}}$$

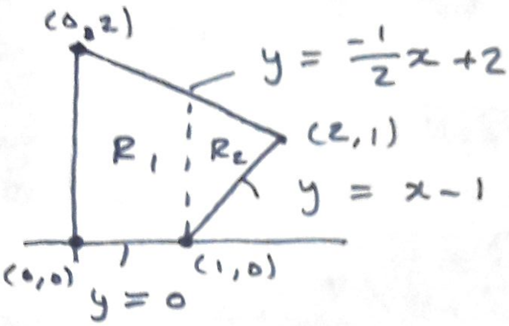
$\Rightarrow$ $(1,3)$ is only Lagrange Point $\Rightarrow$ $(1,3)$ minimizes function.

2. (25 points) Determine the following double integral

$$\iint_R x + 2 \, dx dy,$$

where R is the region with corners $(0, 0)$, $(1, 0)$, $(2, 1)$ and $(0, 2)$.

Solution:



$$\iint_{R_1} x+2 \, dx \, dy = \int_0^1 \left(\int_0^{-\frac{1}{2}x+2} x+2 \, dy \right) dx = \int_0^1 (x+2) y \Big|_0^{-\frac{1}{2}x+2} dx$$

$$= \int_0^1 (x+2)\left(\frac{-1}{2}x+2\right) dx = \int_0^1 \frac{-1}{2}x^2 + x + 4 \, dx = \frac{-1}{6}x^3 + \frac{1}{2}x^2 + 4x \Big|_0^1$$

$$= \frac{-1}{6} + \frac{1}{2} + 4 = \frac{14}{3}$$

$$\iint_{R_2} x+2 \, dx \, dy = \int_1^2 \left(\int_{x-1}^{-\frac{1}{2}x+2} x+2 \, dy \right) dx = \int_1^2 (x+2) y \Big|_{x-1}^{-\frac{1}{2}x+2} dx$$

$$= \int_1^2 (x+2)\left(-\frac{3}{2}x+3\right) dx = \int_1^2 \frac{-3}{2}x^2 + 6 \, dx = \frac{-1}{2}x^3 + 6x \Big|_1^2$$

$$= -4 + 12 + \frac{1}{2} - 6 = \frac{5}{2} \Rightarrow \iint_R x+2 \, dx \, dy = \frac{14}{3} + \frac{5}{2} = \frac{43}{6}$$

3. (25 points) Find the general solution to the following differential equation:

$$2(\cos(x) - 4)^2 y' = y^2 \sin(x)$$

Hence, or otherwise, find solutions with the following initial conditions: $y(0) = 0$ and $y(\pi/2) = 2$.

Solution:

$$y' = y^2 \cdot \frac{\sin(x)}{2(\cos(x) - 4)} \Rightarrow y = 0 \text{ only constant solution.}$$

$$\int \frac{1}{y^2}\, dy = \int \frac{\sin(x)}{2(\cos(x)-4)}\, dx$$

$$u = \cos(x) - 4 \Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow dx = \frac{du}{-\sin(x)} \Rightarrow$$

$$\int \frac{\sin(x)}{2(\cos(x)-4)}\, dx = \int \frac{-1}{2u}\, du = \frac{-1}{2}\ln|u| + C = \frac{-1}{2}\ln|\cos(x)-4| + C$$

$$\Rightarrow \frac{-1}{y} = \frac{-1}{2}\ln|\cos(x) - 4| + C$$

$$\Rightarrow y = \begin{cases} \dfrac{1}{\frac{1}{2}\ln|\cos(x)-4| - C} \\ 0 \end{cases} \quad C \text{ any constant}$$

$$y(0) = 0 \Rightarrow y(x) = 0 \quad \text{constant solution}$$

$$y\left(\frac{\pi}{2}\right) = 2 \Rightarrow \frac{1}{\frac{1}{2}\ln(4) - C} = 2 \Rightarrow -C = \frac{1 - \ln(4)}{2}$$

$$\Rightarrow y(x) = \frac{1}{\frac{1}{2}\ln|\cos(x) - 4| + \frac{1 - \ln(4)}{2}}$$

4. A population grows exponentially with growth constant 2. The emigration rate (as a function of time) is $t^2 + 4$. The initial population has size 20000. What will the population be at time t in the future?

Solution:

$y(t) =$ population size at time t

$$\Rightarrow y' = 2y - t^2 - 4 \Rightarrow y' + \underbrace{(-2)}_{a(t)} y = \underbrace{-t^2 - 4}_{b(t)}$$

$$A(t) = -2t \Rightarrow e^{A(t)} = e^{-2t} \Rightarrow$$

$$y(t) = \frac{1}{e^{-2t}} \cdot \left(\int e^{-2t} \times (-(t^2+4))\, dt \right)$$

$$= -e^{2t} \left(\int t^2 e^{-2t} dt + \int 4 e^{-2t} dt \right)$$

$$\int \underset{f(t)}{t^2} \underset{g(t)}{e^{-2t}} dt = -\frac{t^2}{2} e^{-2t} + \frac{2}{2} \int t e^{-2t} dt$$

$$f'(t) = 2t, \quad G(t) = \frac{1}{-2} e^{-2t}$$

$$\int \underset{f(t)}{t} \underset{g(t)}{e^{-2t}} dt = \frac{-t}{2} e^{-2t} + \frac{1}{2} \int e^{-2t} dt = \frac{-t}{2} e^{-2t} - \frac{1}{4} e^{-2t} + C$$

$$f'(t) = 1 \quad G(t) = \frac{1}{-2} e^{-2t}$$

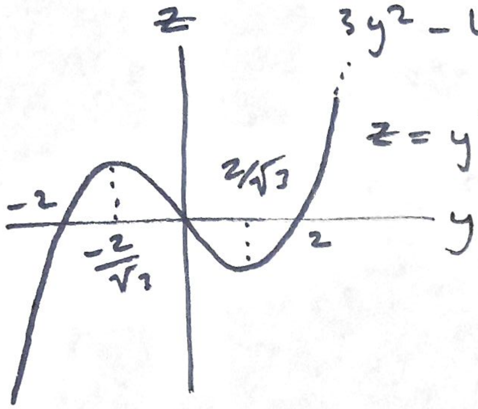
$$\Rightarrow y(t) = -e^{2t} \left(-\frac{t^2}{2} e^{-2t} - \frac{t}{2} e^{-2t} - \frac{1}{4} e^{-2t} - 2 e^{-2t} + C \right)$$

5. (25 points) Consider the differential equation $y' = y^3 - 4y$. Graph the solutions with the following initial conditions: $y(0) = -1$, $y(0) = 0$ and $y(0) = 3/2$. Be sure to label each solution carefully.

Solution:

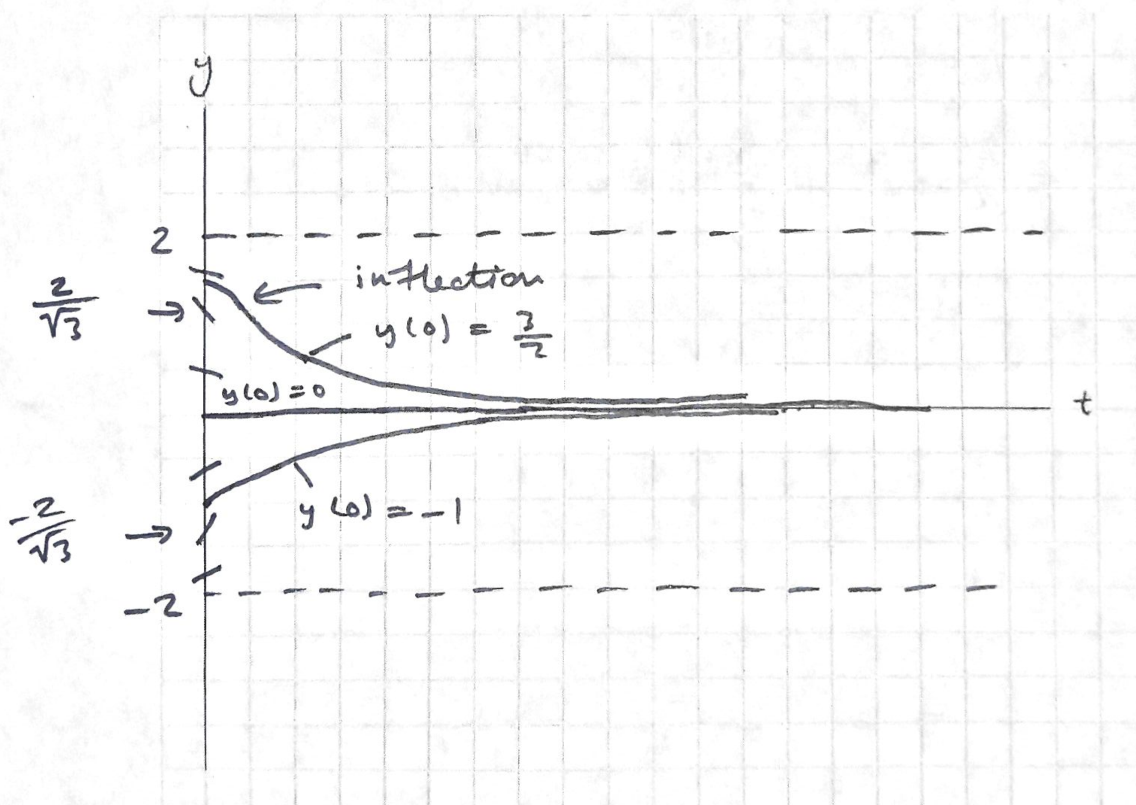
$$y^3 - 4y = 0 \Rightarrow y = 0, 2, -2$$

$$3y^2 - 4 = 0 \Rightarrow y = \pm \frac{2}{\sqrt{3}}$$



$$z = y^3 - 4y \qquad 1 < \frac{2}{\sqrt{3}} < \frac{3}{2}$$

$$\frac{-2}{\sqrt{3}} < -1$$



6. (a) (15 points) Using the integral test, determine whether

$$\sum_{k=3}^{\infty} \frac{1}{k(\ln(k))^2}$$

is convergent or divergent. Be sure to check all the hypotheses of the integral test.

Solution:

$$f(x) = \frac{1}{x(\ln(x))^2} \Rightarrow \quad 1/\ f(k) = \frac{1}{k(\ln(k))^2} \text{ for } k \geq 3 \text{ integer}$$

$$2/\ f'(x) = \frac{-\left((\ln(x))^2 + 2\ln(x)\right)}{\left(x(\ln(x))^2\right)^2} < 0$$

on $[3, \infty)$

$\Rightarrow f(x)$ decreasing on $[3, \infty)$

$\Rightarrow$ Can apply integral test.

$$u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du \Rightarrow$$

$$\int \frac{1}{x(\ln(x))^2}\,dx = \int \frac{1}{u^2}\,du = \frac{-1}{u} + C = \frac{-1}{\ln(x)} + C$$

$$\Rightarrow \int_3^t \frac{1}{x(\ln(x))^2}\,dx = \left.\frac{-1}{\ln(x)}\right|_3^t = \frac{1}{\ln(3)} - \frac{1}{\ln(t)}$$

$$\Rightarrow \int_3^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \lim_{t\to\infty} \frac{1}{\ln(3)} - \frac{1}{\ln(t)} = \frac{1}{\ln(3)} \Rightarrow \text{conv.}$$

$$\Rightarrow \sum_{k=3}^{\infty} \frac{1}{k(\ln(k))^2} \text{ conv.}$$

(b) (10 points) Using this, or otherwise, prove that the the infinite series

$$\sum_{k=3}^{\infty} \frac{1}{k^2(\ln(k))^2}$$

is convergent.

Solution:

$$0 < \frac{1}{k^2(\ln(k))^2} \leq \frac{1}{k(\ln(k))^2} \quad \text{for all } k \geq 3$$

$$\Rightarrow \sum_{k=3}^{\infty} \frac{1}{k^2(\ln(k))^2} \quad \text{conv. by comparison test.}$$

7. (25 points) Determine the Taylor series of the following functions. Write at least the first five non-zero terms of the sum to illustrate the general pattern.

(a)
$$f(x) = e^{-x^2}$$

Solution:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\Rightarrow e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} \dots$$

(b)
$$f(x) = \frac{2}{(1-x)^3}$$

Solution:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

derivative

$$\Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

derivative

$$\Rightarrow \frac{2}{(1-x)^3} = 2 + 3 \cdot 2 \cdot x + 4 \cdot 3 \cdot x^2 + 5 \cdot 4 \cdot x^3 + \dots$$

8. (25 points) For what value of k is the function

$$f(x) = 4xe^{-x^2}$$

a probability density function on $[k, \infty)$?

Solution: Let $u = -x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow dx = \frac{du}{-2x}$

$$\Rightarrow \int 4xe^{-x^2}\,dx = -2\int e^u\,du = -2e^u + C = -2e^{-x^2} + C$$

$$\Rightarrow \int_k^t 4xe^{-x^2}\,dx = -2e^{-x^2}\Big|_k^t = 2e^{-k^2} - 2e^{-t^2}$$

$$\Rightarrow \int_k^\infty 4xe^{-x^2}\,dx = \lim_{t\to\infty} 2e^{-k^2} - 2e^{-t^2} = 2e^{-k^2}$$

1/ $f(x) \geq 0$ on $[k, \infty) \Rightarrow k \geq 0$

2/ $\int_k^\infty f(x)\,dx = 1 \Rightarrow 2e^{-k^2} = 1 \Rightarrow e^{k^2} = 2 \Rightarrow k = \pm\sqrt{\ln(2)}$

By 1/ we know $k = \sqrt{\ln(2)}$ is the only possible value.

9. (25 points) Let X be a discrete random variable which can take value any non-negative integer $0, 1, 2, 3, \ldots\ldots$ Assume that

$$Pr(X = n) = \frac{4}{5^{n+1}},$$

for any n, a non-negative integer. Calculate the probability that X is odd and less than 10. Simplify your answer. You do not need to calculate high powers of 5. What is the variance of X?

Solution:

$$Pr(X = n) = \frac{4}{5^{n+1}} = \left(\frac{1}{5}\right)^n \cdot \left(\frac{4}{5}\right) \quad \text{geometric with } p = \frac{1}{5}$$

$$Pr(X \text{ odd an } < 10) = Pr(X=1) + Pr(X=3) + Pr(X=5) + Pr(X=7) + Pr(X=9)$$

$$= \left(\frac{1}{5}\right)\cdot\left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)^5\cdot\left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)^7\left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)^9\left(\frac{4}{5}\right) \quad a = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) \quad r = \left(\frac{1}{5}\right)^2$$

$$\left(a + ar + ar^2 + ar^3 + ar^4 = \frac{a(1-r^5)}{1-r}\right)$$

$$\Rightarrow Pr(X \text{ odd and } < 10) = \frac{\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)\left(1 - \left(\left(\frac{1}{5}\right)^2\right)^5\right)}{1 - \left(\frac{1}{5}\right)^2}$$

$$Var(X) = \frac{p}{(1-p)^2} = \frac{\frac{1}{5}}{\left(\frac{4}{5}\right)^2} = \frac{1}{16}$$

10. (25 points) Let X be an exponential random variable. Assume that the median of X is $\ln(2)$. What is the expected value of X? Calculate $Pr(X \geq 1)$.

Solution:

$$\int_0^{\ln(2)} k e^{-kx}\, dx = \frac{1}{2} \Rightarrow -e^{-kx}\Big|_0^{\ln(2)} = \frac{1}{2}$$

$$\Rightarrow 1 - e^{-k\ln(2)} = \frac{1}{2} \Rightarrow e^{-k\ln(2)} = \frac{1}{2}$$

$$\Rightarrow k\ln(2) = \ln(2) \Rightarrow k = 1$$

$$\Rightarrow \mathbb{E}(X) = \frac{1}{1} = 1$$

$$Pr(X \geq 1) = \int_1^{\infty} e^{-x}\, dx = \lim_{t\to\infty} \int_1^{t} e^{-x}\, dx$$

$$= \lim_{t\to\infty} -e^{-t} + e^{-1} = e^{-1}$$

END OF EXAM