DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Formulae

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) Find the points (x, y) which minimize the function $\sqrt{6x^2 + y^2}$, subject to the constraint $2x+y = 5$. You may assume, without justification that a minimum exists.

   Solution:

2. (25 points) Determine the following double integral

$$\iint_R x + 2 \, dx dy,$$

where R is the region with corners $(0,0)$, $(1,0)$, $(2,1)$ and $(0,2)$.

Solution:

3. (25 points) Find the general solution to the following differential equation:

$$2(\cos(x) - 4)^2 y' = y^2 \sin(x)$$

Hence, or otherwise, find solutions with the following initial conditions: $y(0) = 0$ and $y(\pi/2) = 2$.

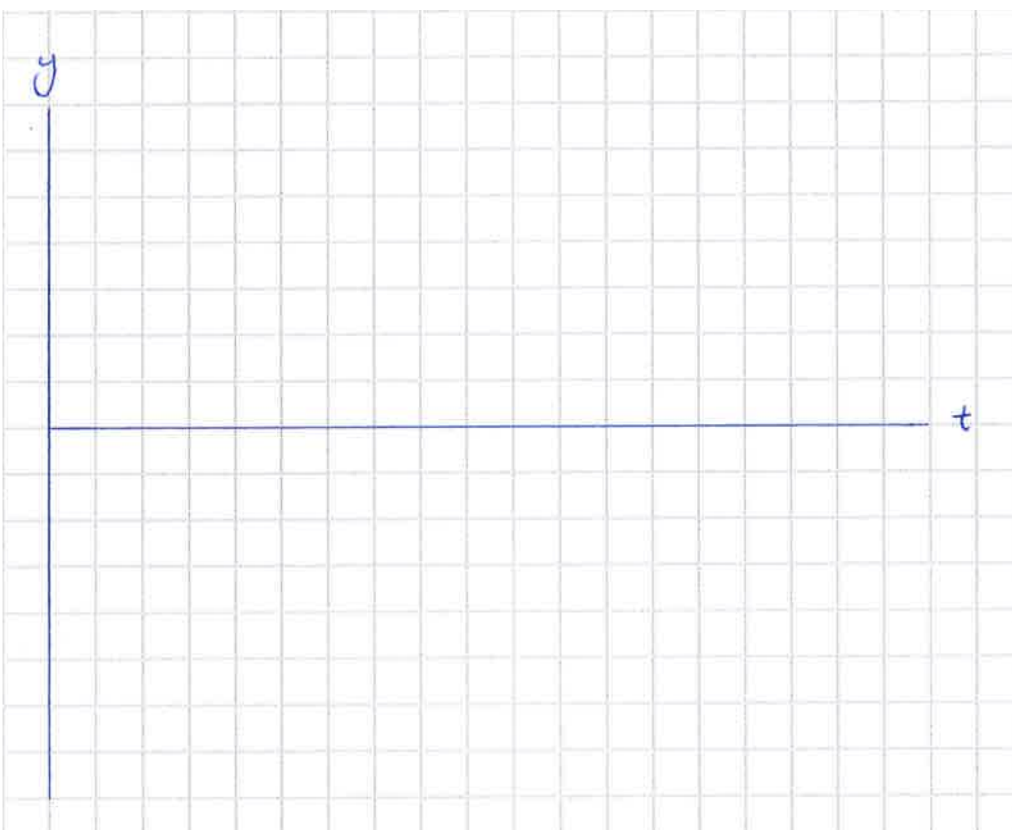
Solution:

4. A population grows exponentially with growth constant 2. The emigration rate (as a function of time) is $t^2 + 4$. The initial population has size 20000. What will the population be at time t in the future?

   Solution:

5. (25 points) Consider the differential equation $y' = y^3 - 4y$. Graph the solutions with the following initial conditions: $y(0) = -1$, $y(0) = 0$ and $y(0) = 3/2$. Be sure to label each solution carefully.

Solution:



6. (a) (15 points) Using the integral test, determine whether

$$\sum_{k=3}^{\infty} \frac{1}{k(\ln(k))^2}$$

is convergent or divergent. Be sure to check all the hypotheses of the integral test.

Solution:

(b) (10 points) Using this, or otherwise, prove that the the infinite series

$$\sum_{k=3}^{\infty} \frac{1}{k^2(\ln(k))^2}$$

is convergent.

Solution:

7. (25 points) Determine the Taylor series of the following functions. Write at least the first five non-zero terms of the sum to illustrate the general pattern.

(a)

$$f(x) = e^{-x^2}$$

Solution:

(b)

$$f(x) = \frac{2}{(1-x)^3}$$

Solution:

8. (25 points) For what value of k is the function

$$f(x) = 4xe^{-x^2}$$

a probability density function on $[k, \infty)$?

Solution:

9. (25 points) Let X be a discrete random variable which can take value any non-negative integer $0, 1, 2, 3, ...$ Assume that

$$Pr(X = n) = \frac{4}{5^{n+1}},$$

for any n, a non-negative integer. Calculate the probability that X is odd and less than 10. Simplify your answer. You do not need to calculate high powers of 5. What is the variance of X?

Solution:

10. (25 points) Let X be an exponential random variable. Assume that the median of X is $\ln(2)$. What is the expected value of X? Calculate $Pr(X \geq 1)$.

Solution:

END OF EXAM