This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) Find the points (x, y) which maximise the function y^2x, subject to the constraint $y^2 + 4xy = 1200$. You may assume, without justification, that a maximum exists.

Solution:

$$F(x,y,\lambda) = y^2x + \lambda y^2 + 4\lambda xy - 1200\lambda$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = y^2 + 4\lambda y = 0 \\ \dfrac{\partial F}{\partial y} = 2xy + 2\lambda y + 4\lambda x = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda = \dfrac{-y}{4} \\ \lambda = \dfrac{-2xy}{2y+4x} \end{array} \Rightarrow \frac{-y}{4} = \frac{-2xy}{2y+4x}$$

$$\frac{\partial F}{\partial \lambda} = y^2 + 4xy - 1200 = 0$$

$$\Rightarrow 2y^2 + 4xy = 8xy$$

$$\Rightarrow x = \frac{y}{2}$$

$$\Rightarrow y^2 + 4\left(\frac{y}{2}\right)y - 1200 = 0$$

$$\Rightarrow 3y^2 - 1200 = 0 \Rightarrow y^2 = 400 \Rightarrow y = 20 \text{ or } -20$$

$$y = 20 \Rightarrow x = 10 \Rightarrow \lambda = -5$$

$$y = -20 \Rightarrow x = -10 \Rightarrow \lambda = 5$$

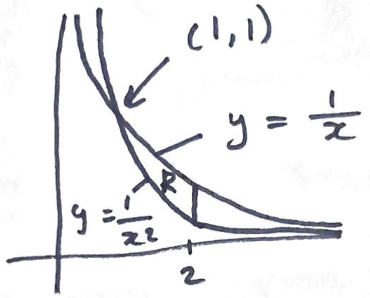
$\Rightarrow$ $(10, 20)$ and $(-10, -20)$ are Lagrange points

$20^2 \cdot 10 > (-20)^2(-10) \Rightarrow (10, 20)$ is where function is maximized.

2. (25 points) Let R be the region enclosed by $y = 1/x^2$, $y = 1/x$ and $x = 2$ in the first quadrant. Determine the following double integral

$$\iint_R \frac{x}{y}\, dxdy$$

Solution:



$$\left(\frac{1}{x^2} = \frac{1}{x} \underset{(x \neq 0)}{\Rightarrow} x^2 = x \Rightarrow 1 \right)$$

$$\iint_R \frac{x}{y}\, dx\, dy = \int_1^2 \left(\int_{\frac{1}{x^2}}^{\frac{1}{x}} \frac{x}{y}\, dy \right) dx = \int_1^2 x \ln(y) \Big|_{x^{-2}}^{x^{-1}} dx$$

$$= \int_1^2 \underbrace{x \ln(x^{-1})}_{= -x\ln(x)} - \underbrace{x \ln(x^{-2})}_{= -2x\ln(x)}\, dx = \int_1^2 \underset{g(x)}{x}\, \underset{f(x)}{\ln(x)}\, dx$$

$$f(x) \Rightarrow f'(x) = \frac{1}{x}$$

$$G(x) = \frac{x^2}{2}$$

$$\int x \ln(x)\, dx = \frac{x^2}{2} \ln(x) - \int \frac{x}{2}\, dx = \frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C$$

$$\Rightarrow \iint_R \frac{x}{y}\, dx\, dy = \frac{x^2}{2} \ln(x) - \frac{x^2}{4} \Big|_1^2 = 2\ln(2) - 1$$

$$- \left(0 - \frac{1}{4} \right)$$

$$= 2\ln(2) - \frac{3}{4}$$

3. (25 points) Find the general solution to the following differential equation:

$$x^2 y' = y\cos(1/x) + y$$

Hence, or otherwise, find solutions with the following initial conditions: $y(1/\pi) = 0$ and $y(1/\pi) = -1$.

Solution:

$$y' = y\left(x^{-2}\cos(1/x) + x^{-2}\right) \Rightarrow y = 0 \text{ is only constant solution}$$

$$\int \frac{1}{y}\,dy = \int x^{-2}\cos\left(\frac{1}{x}\right) + x^{-2}\,dx = \int x^{-2}\cos(x^{-1})\,dx + \int x^{-2}\,dx$$

$$u = x^{-1} \Rightarrow \frac{du}{dx} = -x^{-2} \Rightarrow dx = \frac{du}{-x^{-2}} \Rightarrow$$

$$\int x^{-2}\cos(x^{-1})\,dx = -\int \cos(u)\,du = -\sin(u) + C = -\sin(1/x) + C$$

$$\Rightarrow \ln|y| = -\sin\left(\frac{1}{x}\right) - \frac{1}{x} + C$$

$$\Rightarrow y = \begin{cases} \pm e^C \cdot e^{-\sin(\frac{1}{x}) - \frac{1}{x}} & \pm e^C \text{ any non-zero constant} \\ 0 \end{cases}$$

$y(\frac{1}{\pi}) = 0 \Rightarrow y(x) = 0$ the constant solution

$$y\left(\frac{1}{\pi}\right) = -1 \Rightarrow \pm e^C e^{-\pi} = -1 \Rightarrow \pm e^C = -e^{\pi}$$

$$\Rightarrow y(x) = -e^{\pi} \cdot e^{\left(-\sin(\frac{1}{x}) - \frac{1}{x}\right)}$$

4. (25 points) A company expects to have a constant income rate for the next ten years. They will invest it in a saving account with 50% annual interest rate. If they want to have \$10000000 in the account after ten years, what must their income rate be? You do not need to simplify your answer.

Solution:

k = constant annual interest rate

Accumulated Value over $[0,10]$ $= \int_0^{10} k e^{\frac{1}{2}(10-t)} \, dt = 10000000$

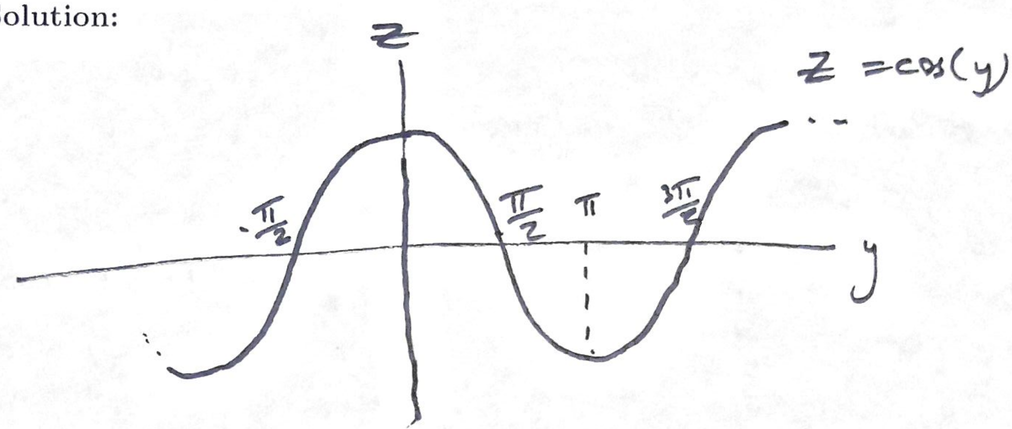
$$\Rightarrow \quad ke^5 \int_0^{10} e^{-\frac{1}{2}t} \, dt = ke^5 \left(-2e^{-\frac{1}{2}t} \Big|_0^{10} \right) = 10000000$$

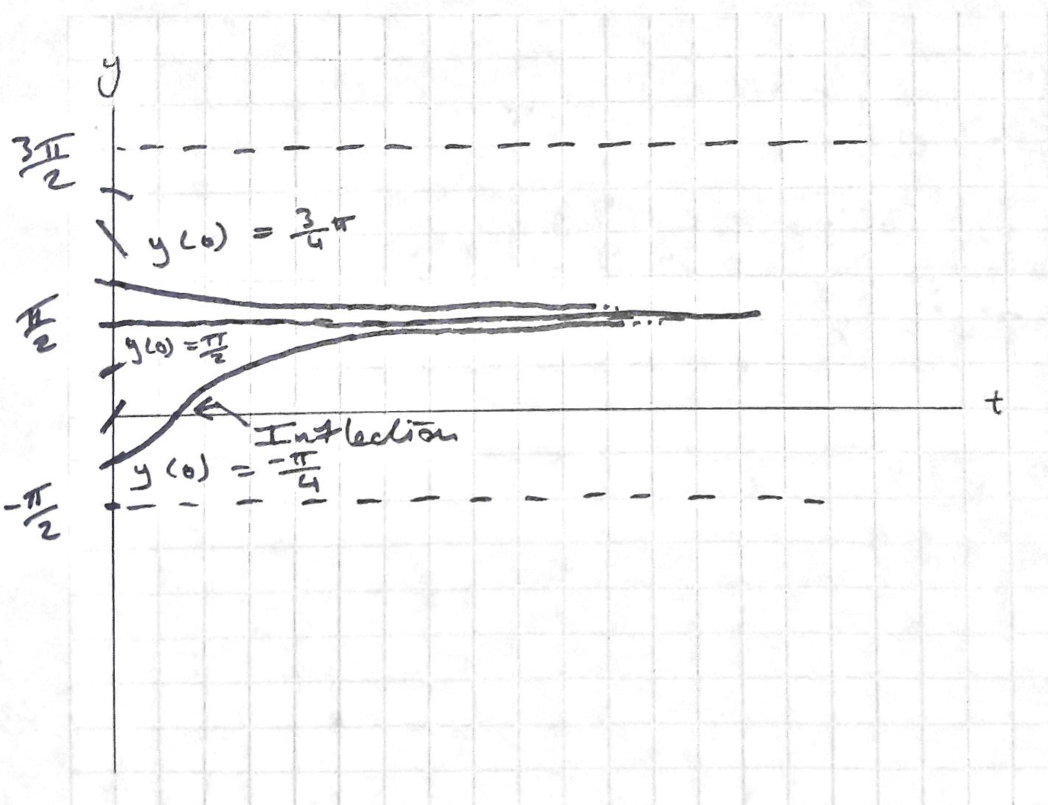
$$\Rightarrow \quad ke^5 \left(-2e^{-5} + 2 \right) = 10000000$$

$$\Rightarrow \quad k = \frac{10000000}{2e^5 - 2} \text{ \$ per year}$$

5. (25 points) Consider the differential equation $y' = \cos(y)$. Graph the solutions with the following initial conditions: $y(0) = -\pi/4$, $y(0) = \pi/2$ and $y(0) = 3\pi/4$. Be sure to label each solution carefully.

Solution:





6. (a) (15 points) Using the integral test, determine whether

$$\sum_{k=1}^{\infty} \frac{1}{k^3}$$

is convergent or divergent. Be sure to check all the hypotheses of the integral test.

Solution:

$$f(x) = \frac{1}{x^3} \Rightarrow \begin{array}{l} 1/\ f(k) = \frac{1}{k^3} \text{ for all } k \geq 1 \\ 2/\ f'(x) = \frac{-3}{x^4} < 0 \text{ on } [1, \infty) \end{array}$$

$\Rightarrow f(x)$ decreasing on $[1,\infty)$

$\Rightarrow$ Can apply integral test

$$\int \frac{1}{x^3}\,dx = \frac{1}{-2x^2} + C$$

$$\Rightarrow \int_1^t \frac{1}{x^3}\,dx = \left.\frac{1}{-2x^2}\right|_1^t = \frac{1}{2} - \frac{1}{2t^2}$$

$$\Rightarrow \int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \frac{1}{2} - \frac{1}{2t^2} = \frac{1}{2} \Rightarrow \text{converged}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^3} \text{ convergent}$$

(b) (10 points) Using this, or otherwise, determine if the infinite series

$$\sum_{k=1}^{\infty} \frac{\sin^2(k)}{k^3}$$

is convergent or divergent.

Solution:

$0 \leq \sin^2(k) \leq 1$ for all $k \geq 1$

$\Rightarrow \quad 0 \leq \dfrac{\sin^2(k)}{k^3} \leq \dfrac{1}{k^3}$ for all $k \geq 1$

$\Rightarrow \quad \displaystyle\sum_{k=1}^{\infty} \frac{\sin^2(k)}{k^3}$ convergent by comparison test.

7. (25 points) Determine the Taylor series of the following functions. Write at least the first five non-zero terms of the sum to illustrate the general pattern.

(a)

$$f(x) = e^{\frac{x}{2}}$$

Solution:

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\Rightarrow e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} + \dots$$

(b)

$$f(x) = \ln(1 - 2x^2)$$

Solution:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

$$\Rightarrow \ln(1-2x^2) = \ln(1+(-2x^2))$$

$$= (-2x^2) - \frac{(-2x^2)^2}{2} + \frac{(-2x^2)^3}{3} - \frac{(-2x^2)^4}{4} + \frac{(-2x^2)^5}{5} - \dots$$

$$= -2x^2 - \frac{2^2x^4}{2} - \frac{2^3x^6}{3} - \frac{2^4x^8}{4} - \frac{2^5x^{10}}{5} - \dots$$

8. (25 points) Let X be a continuous random variable on $[0, \pi/2]$ with probability density function $f(x) = \sin(x)$. Calculate the cumulative distribution function of X. Calculate $Var(X)$.

C.D.F.

$$\int \sin(x)\,dx = -\cos(x) + C = F(x)$$

$$F(0) = 0 \Rightarrow -\cos(0) + C = 0 \Rightarrow C = 1$$

$$\Rightarrow F(x) = 1 - \cos(x)$$

$$\mathbb{E}(X) = \int_0^{\pi/2} \underset{f(x)}{x}\,\underset{g(x)}{\sin(x)}\,dx = -x\cos(x) + \sin(x)\Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

$f'(x) = 1$ $G(x) = -\cos(x)$

$$\int \underset{f(x)}{x^2}\,\underset{g(x)}{\sin(x)}\,dx = -x^2\cos(x) + 2\int x\cos(x)\,dx$$

$f(x) = x^2$ $g(x) = \sin(x)$

$f'(x) = 2x$ $G(x) = -\cos(x)$

$$\int \underset{f(x)}{x}\,\underset{g(x)}{\cos(x)}\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C$$

$f'(x) = 1$ $G(x) = \sin(x)$

$$\Rightarrow \int_0^{\pi/2} x^2\sin(x)\,dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x)\Big|_0^{\frac{\pi}{2}}$$

$$= \pi - 2$$

$$\Rightarrow Var(X) = (\pi - 2) - 1^2 = \pi - 3$$

9. (25 points) Let Z be the standard normal random variable. Assume that

$$Pr(|Z| \leq \frac{1}{2}) = 0.4.$$

If X is a normal random variable with variance 4 and such that $Pr(X \geq 1) = \frac{1}{2}$, calculate

$$Pr(X \leq 2).$$

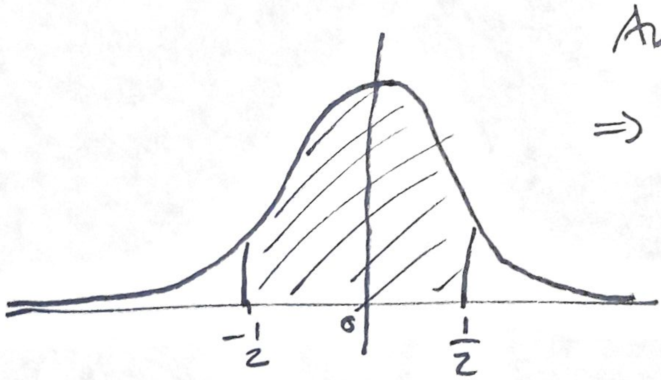
Solution:

$Var(X) = 4 = \sigma^2 \Rightarrow \sigma = 2$

$Pr(X \geq 1) = \frac{1}{2} \Rightarrow \text{Median}(X) = 1 \Rightarrow \mu = 1$

$Pr(X \leq 2) = Pr(Z \leq \frac{2-1}{2}) = Pr(Z \leq \frac{1}{2})$

Standard Normal.



Area(▨) $= Pr(|Z| \leq \frac{1}{2}) = 0.4$

$\Rightarrow Pr(0 \leq Z \leq \frac{1}{2}) = \frac{0.4}{2} = 0.2$

$Pr(Z \leq \frac{1}{2}) = \underbrace{Pr(Z \leq 0)}_{= \frac{1}{2}} + \underbrace{Pr(0 \leq Z \leq \frac{1}{2})}_{= 0.2} = 0.7$

$\Rightarrow Pr(X \leq 2) = 0.7$

10. (25 points) Let X be a discrete random variable which can take value any non-negative integer $0, 1, 2, 3, \ldots\ldots$ Assume that

$$Pr(X = n) = \frac{2}{3^{n+1}},$$

for any n, a non-negative integer. Calculate the probability that X is even. What is the expected value of X?

Solution:

$$Pr(X=n) = \left(\frac{1}{3}\right)^n \cdot \frac{2}{3} \leftarrow \text{geometric with } p = \frac{1}{3}$$

$$Pr(X \text{ even}) = Pr(X=0) + Pr(X=2) + P_r(X=4) + \cdots$$

$$= \frac{2}{3} + \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right) + \cdots$$

geometric with $a = \frac{2}{3}$, $r = \left(\frac{1}{3}\right)^2$

$$\Rightarrow Pr(X \text{ even}) = \frac{\frac{2}{3}}{1 - \left(\frac{1}{3}\right)^2}$$

$$E(X) = \frac{p}{1-p} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

END OF EXAM