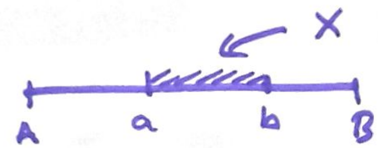
Quick Review

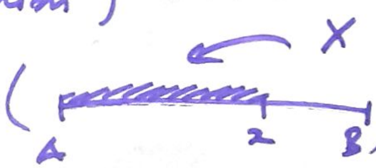
X – C.R.V. on $[A, B]$

Given $[a, b]$ in $[A, B]$, how do we calculate $Pr(a \leq X \leq b)$?

$Pr(a \leq X \leq b) = Pr($  $)$

Two approaches :

C.D.F. (Cumulative Distribution Function)

$= F(x) = Pr(X \leq x) = Pr($  $)$

$\Rightarrow Pr(a \leq X \leq b) = F(b) - F(a)$

Fact : $F(A) = 0$

P.D.F. (Probability Density Function) is function $f(x)$ on $[A, B]$ such that $Pr(a \leq X \leq b) = \int_a^b f(x)\,dx$

Fact $f(x) = F'(x)$

$\mathbb{E}(X) = \int_A^B x f(x)\,dx$, $Var(X) = \int_A^B x^2 f(x)\,dx - (\mathbb{E}(X))^2$

Exponential Random Variables

X a C.R.V. on $[0,\infty)$ is exponential if P.D.F. is of form $f(x) = ke^{-kx}$ for some constant $k > 0$.

1/ $k > 0 \Rightarrow ke^{-kx} > 0$ on $[0,\infty)$

2/ $\displaystyle\int_0^\infty ke^{-kx}\,dx = \lim_{t\to\infty}\int_0^t ke^{-kx}\,dx = \lim_{t\to\infty} -e^{-kx}\Big|_0^t$

$$= \lim_{t\to\infty} -e^{-kt} - (-1) = 1$$

Let's calculate expected value and variance.

$$\int x f(x)\,dx = \int x\,ke^{-kx}\,dx.$$

I.B.P. $\quad f(x) = kx \quad f'(x) = k$
$\qquad g(x) = e^{-kx} \quad G(x) = \frac{-1}{k}e^{-kx} \quad \Rightarrow$

$$\int x k e^{-kx}\,dx = -xe^{-kx} + \int e^{-kx}\,dx = -xe^{-kx} - \frac{1}{k}e^{-kx}$$

$$= \frac{-x}{e^{kx}} - \frac{1}{k}\cdot\frac{1}{e^{kx}}$$

$$\Rightarrow \int_0^t x k e^{-kx}\,dx = \frac{-x}{e^{kx}} - \frac{1}{k}\cdot\frac{1}{e^{kx}}\Big|_0^t$$

$$= \frac{-t}{e^{kt}} - \frac{1}{k}\cdot\frac{1}{e^{kt}} + \frac{1}{k}$$

(Recall $\lim_{t\to\infty} \frac{t}{e^{kt}} = 0$ and $\lim_{t\to\infty} \frac{1}{e^{kt}} = 0$)

$$\Rightarrow \int_0^\infty x k e^{-kx}\,dx = \frac{1}{k} \Rightarrow \mathbb{E}(X) = \frac{1}{k}.$$

We can do a similar calculation to show $Var(X) = \frac{1}{k^2}$

Conclusions

X an exponential C.R.V. on $[0,\infty) \Rightarrow$

$f(x) = ke^{-kx}$ for some $k>0$.

$\mathbb{E}(X) = \frac{1}{k}$, $Var(X) = \frac{1}{k^2}$

Example The lifetime of a light bulb is an exponential C.R.V. The expected lifetime of a lightbulb is 100 days

1/ Find the probability density function?

2/ What is the cumulative distribution function?

3/ What is the probability a lightbulb lasts for more than 40 days.

1/ X exponential $\Rightarrow f(x) = ke^{-kx}$ for some $k > 0$

$$E(X) = \frac{1}{k} = 100 \Rightarrow k = \frac{1}{100}$$

$$\Rightarrow f(x) = \frac{1}{100} e^{-\frac{1}{100}x}$$

2/ $\int f(x)\,dx = -e^{-\frac{1}{100}x} + C = F(x)$

Recall $F(0) = 0 \Rightarrow C = 1$

$$\Rightarrow F(x) = 1 - e^{-\frac{1}{100}x}$$

3/ $Pr(X \geq 40) = \int_{40}^{\infty} \frac{1}{100} e^{-\frac{1}{100}x} dx = \lim_{t \to \infty} -e^{-\frac{1}{100}x} \Big|_{40}^{t}$

$$= \lim_{t \to \infty} \left(-e^{-\frac{1}{100}t} - (-e^{-0.4}) \right) = e^{-0.4} \approx 0.67032$$

Example

The wait time for a train is an <u>exponential</u> random variable. If the expected wait time is 1 hour, what is the median wait time?

$\mathbb{E}(X) = 1 = \frac{1}{k} \Rightarrow k = 1 \Rightarrow f(x) = e^{-x}$

Need $M \geq 0$ such that $\Pr(X \leq M) = \frac{1}{2} \Rightarrow \int_0^M e^{-x}\,dx = \frac{1}{2}$

$\Rightarrow -e^{-x}\Big|_0^M = \frac{1}{2} \Rightarrow -e^{-M} + 1 = \frac{1}{2} \Rightarrow e^{-M} = \frac{1}{2}$

$\Rightarrow e^{M} = 2 \Rightarrow M = \ln(2)$ hours

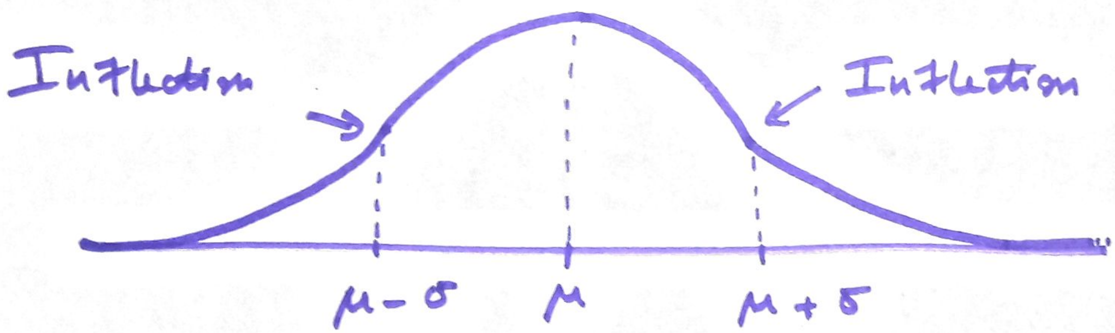
Normal Random Variables

X, a C.R.V. on $(-\infty, \infty)$ is <u>normal</u> if

$$\text{P.D.F} = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \left(\begin{array}{l} \mu, \sigma \text{ constant} \\ \sigma > 0 \end{array} \right)$$

Needed to make a P.D.F.

Basic Graph :



Often called a "bell curve".

Many familiar random variables are normal.

Examples

1/ Life span in USA.

2/ Adult height in USA

3/ Shoe size

4/ Lead levels in hair

Maybe not so familiar !

<u>Facts</u> X <u>normal</u> with parameters μ and $\sigma > 0$

$$\Rightarrow \left.\begin{array}{l} \mathbb{E}(X) = \mu \\ \text{Median}(X) = \mu \\ \text{Var}(X) = \sigma^2 \end{array}\right\} \text{Not easy to show. Just take them as } \underline{\text{facts}}.$$

X = <u>standard normal</u> C.R.V. if $\mu = 0$ and $\sigma = 1$

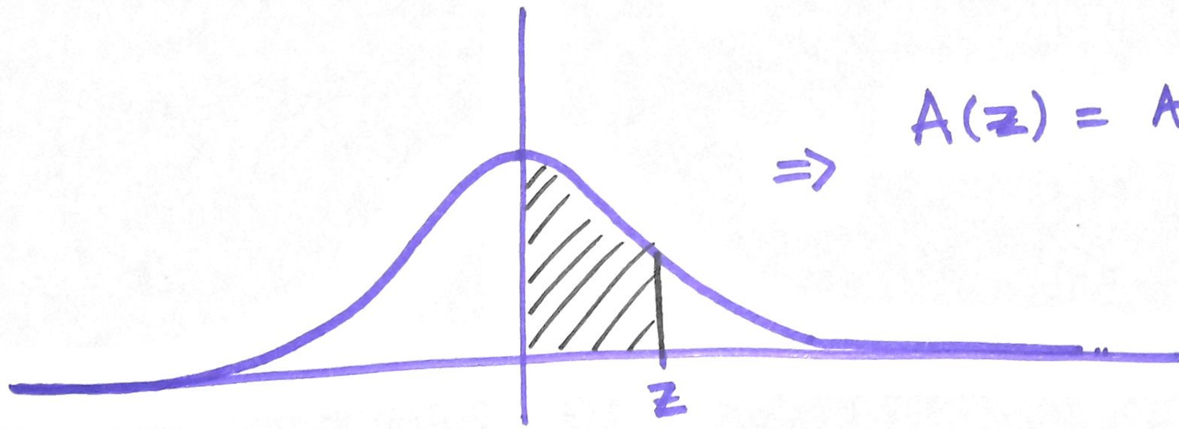
$$\Rightarrow \text{P.D.F.} = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Rightarrow \Pr(a \le x \le b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

<u>Problem</u> : <u>Cannot</u> integrate $e^{-\frac{x^2}{2}}$ by elementary means. We could in theory use Taylor Series but that is itself quite cumbersome, and would give probabilities as infinite series.

Practical Solution : Give areas under standard normal curve in a table (Standard Normal Table)

For $z \geqslant 0$ let $A(z) = \int_0^z \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \Rightarrow$



$\Rightarrow A(z) = \text{Area}$ (▨)

$F(z) = A(z) + \frac{1}{2} = Pr(X \leq z)$

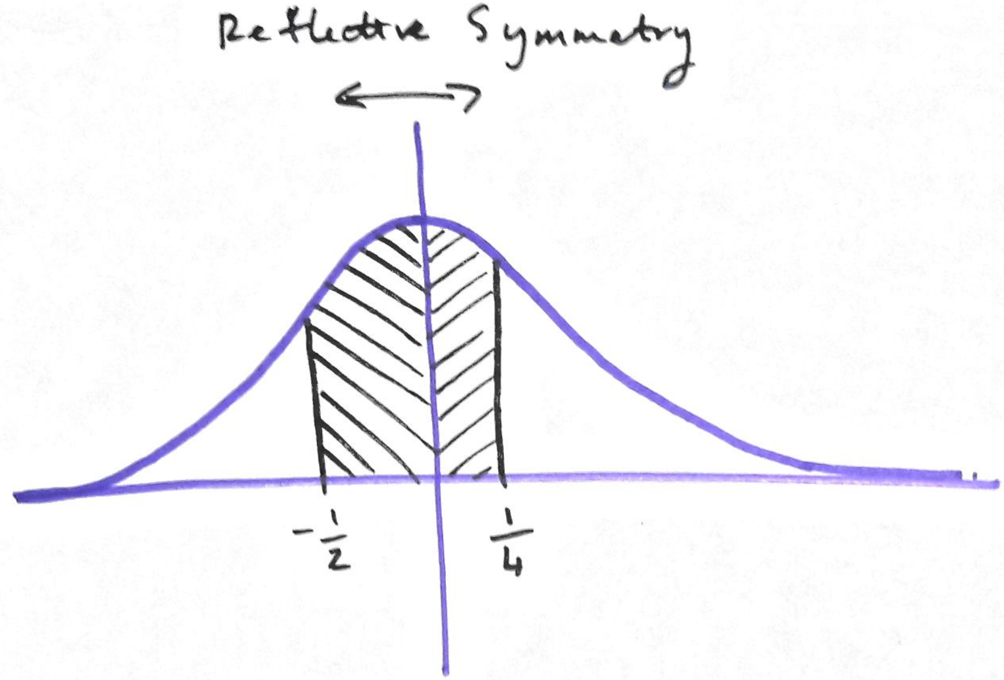
C.D.F.

area under negative part of curve

The values of $A(z)$ are given on P 543.

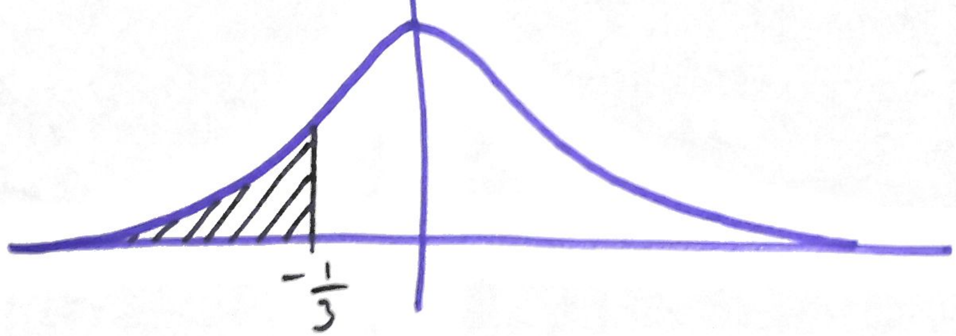
<u>Example</u> X standard normal C.R.V.

$Pr\left(\frac{-1}{2} \leq X \leq \frac{1}{4}\right) = ?$ $\quad Pr\left(X \leq -\frac{1}{3}\right) = ?$



$$\Pr\left(-\tfrac{1}{2} \le X \le \tfrac{1}{4}\right) = \text{Area}(\ ) + \text{Area}(\ ) = A\left(\tfrac{1}{2}\right) + A\left(\tfrac{1}{4}\right)$$

$$= 0.1915 + 0.0987 = 0.2902$$



$$\Pr\left(X \le -\tfrac{1}{3}\right) = \tfrac{1}{2} - A\left(\tfrac{1}{3}\right) = 0.5 - 0.1293 = 0.3707$$

Can we use the standard Normal Table to determine probabilities of non-standard normal random variables?

X - normal C.R.V. with parameters μ and $\sigma > 0$

Z - standard normal C.R.V.

$$Pr(X \le x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

$$u = \frac{t-\mu}{\sigma} \Rightarrow \frac{du}{dt} = \frac{1}{\sigma} \Rightarrow dt = \sigma\, du \Rightarrow$$

$$\int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

Remembering to change limits of integration.

$$\| \qquad\qquad \|$$

$$Pr(X \le x) \qquad\qquad Pr\left(Z \le \frac{x-\mu}{\sigma}\right)$$

Calculate using standard normal table

$$\Rightarrow \quad Pr(a \le X \le b) = Pr\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

Normal with μ and $\sigma > 0$. Standard normal

<u>Example</u>

Historians have collected evidence that suggests President Andrew Jackson suffered from lead poisoning.

Experimental Data : Jackson's hair had a mean lead level of 130.5 ppm

"parts per million".

Todays lead levels follow a normal distribution with mean 10 ppm and variance 25. $\Rightarrow \mu = 10, \sigma = 5$

X = lead level of random person today

very small

$$Pr(X \geq 130.5) = Pr\left(Z \geq \frac{130.5-10}{5}\right) = Pr(Z \geq 24.1) \approx 0$$

By modern standards Jackson had lead poissoning.