Expected Value and Variance

Ten students sit an exam and get the following scores:

50, 60, 60, 70, 70, 90, 100, 100, 100, 100

X = Score of randomly chosen student

X a D.R.V. with possible values 50, 60, 70, 90, 100

$Pr(X=50) = \frac{1}{10}$, $Pr(X=60) = \frac{2}{10}$, $Pr(X=70) = \frac{2}{10}$, $Pr(X=90) = \frac{1}{10}$, $Pr(X=100) = \frac{4}{10}$

Can we express the mean in terms of these probabilities?

$$\text{Mean} = \frac{50+60+60+70+70+90+100+100+100+100}{10} \quad (= 80)$$

$$= \frac{50 \times 1 + 60 \times 2 + 70 \times 2 + 90 \times 1 + 100 \times 4}{10}$$

$$= 50 \cdot Pr(X=50) + 60 \cdot Pr(X=60) + 70 \cdot Pr(X=70) + 90 \cdot Pr(X=90) + 100 \cdot Pr(X=100$$

Can we express the variance in terms of the probabilities?

$$\text{Variance} = \frac{(50-\overset{\text{mean}}{80})^2 + (60-80)^2 + (60-80)^2 + (70-80)^2 + (70-80)^2 + (90-80)^2 + (100-80)^2 + (100-80)^2 + (100-80)^2 + (100-80)^2}{10}$$

$$= (50-80)^2 \, Pr(X=50) + (60-80)^2 \, Pr(X=60) + (70-80)^2 \, Pr(X=70)$$
$$(90-80)^2 \, Pr(X=90) + (100-80)^2 \cdot Pr(X=100)$$

Definition X D.R.V. with possible values $a_1, a_2, a_3, \dots, a_n$

$$\mathbb{E}(X) = a_1 Pr(X=a_1) + a_2 Pr(X=a_2) + \dots + a_n Pr(X=a_n)$$

expected value of X

$$Var(X) = (a_1 - \mathbb{E}(X))^2 \, Pr(X=a_1) + \dots + (a_n - \mathbb{E}(X))^2 \, Pr(X=a_n)$$

variance of X

If X takes possible values $a_1, a_2, a_3, \dots$ infinite sequence.

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} a_k \, Pr(X=a_k)$$

infinite series

$$Var(X) = \sum_{k=1}^{\infty} (a_k - \mathbb{E}(X))^2 \, Pr(X=a_k)$$

Remark Intuitively, if random event is repeat many times the mean tends to $\mathbb{E}(X)$ and the variance tends to $Var(X)$.

Definition X - C.R.V. on $[A, B]$ with P.D.F. $f(x)$

$$\mathbb{E}(X) = \int_A^B x f(x)\, dx$$

continuous versions of previous formulae

$$Var(X) = \int_A^B (x - \mathbb{E}(X))^2 f(x)\, dx$$

We replace these formulae with <u>improper</u> integrals if we replace $[A, B]$ with $[A, \infty)$, $(-\infty, B]$ or $(-\infty, \infty)$

Very Useful Fact :

$$Var(X) = \int_A^B (x - \mathbb{E}(X))^2 f(x)\, dx = \int_A^B (x^2 - 2\mathbb{E}(X)x + \mathbb{E}(X)^2) f(x)\, dx$$

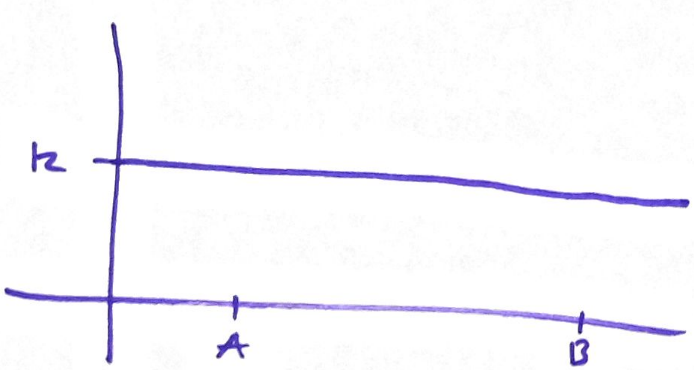
$$= \int_A^B x^2 f(x)\,dx - 2\mathbb{E}(X) \underbrace{\int_A^B x f(x)\,dx}_{\mathbb{E}(X} + (\mathbb{E}(X))^2 \underbrace{\int_A^B f(x)\,dx}_{1}$$

$$\Rightarrow \quad Var(X) = \int_A^B x^2 f(x)\,dx - (\mathbb{E}(X))^2$$

Alternate Formula

<u>Example</u> X C.R.V. on $[A, B]$ is <u>uniform</u> if P.D.F is <u>constant function</u>.

So $f(x) = k$



1/ $f(x) \geq 0 \Rightarrow k \geq 0$

2/ $\int_A^B f(x)\,dx = 1 \Rightarrow k(B-A) = 1$

$\Rightarrow k = \dfrac{1}{B-A}$

$\Rightarrow$ X C.R.V. on $[A, B]$ is <u>uniform</u> if $f(x) = \frac{1}{B-A}$

$$\mathbb{E}(X) = \int_A^B x f(x)\,dx = \frac{1}{B-A}\int_A^B x\,dx = \frac{1}{B-A}\,\frac{x^2}{2}\Big|_A^B$$

$$= \frac{1}{B-A}\cdot\frac{B^2-A^2}{2} = \frac{B+A}{2} \leftarrow \text{midpoint of } [A,B]$$

$$\mathrm{Var}(X) = \int_A^B x^2 f(x)\,dx - \left(\frac{B+A}{2}\right)^2 \leftarrow \text{alternate formula}$$

$$= \frac{1}{B-A}\int_A^B x^2\,dx - \left(\frac{B+A}{2}\right)^2$$

$$= \frac{1}{B-A}\,\frac{B^3-A^3}{3} - \left(\frac{B+A}{2}\right)^2 = \frac{(B-A)^2}{12}$$