Evaluation of Definite Integrals

Fundamental Theorem : $F'(x) = f(x) \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$

General Strategy : 1/ First calculate $\int f(x)\,dx = F(x) + C$

2/ Then calculate $F(b) - F(a)$ $\left(= \int_a^b f(x)\,dx \right)$

Example 1/ $\int_0^1 x(x^2+1)^2\,dx = ?$

Let $u = x^2 + 1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$\Rightarrow \int x(x^2+1)^2\,dx = \int x(x^2+1)^2 \frac{du}{2x} = \frac{1}{2}\int (x^2+1)^2\,du = \frac{1}{2}\int u^2\,du$

$= \frac{1}{6}u^3 + C = \frac{1}{6}(x^2+1)^3 + C$

$\Rightarrow \int_0^1 x(x^2+1)^2\,dx = \frac{1}{6}(x^2+1)^3 \Big|_0^1 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}$

2/ $\int_1^e \frac{\ln(x)}{x^2}\,dx = ?$ $\quad f(x) = \ln(x) \quad g(x) = \frac{1}{x^2}$

$f'(x) = \frac{1}{x} \quad G(x) = \frac{-1}{x}$

$\Rightarrow \int \frac{\ln(x)}{x^2}\,dx = \frac{-\ln(x)}{x} - \int \frac{1}{x} \cdot \left(\frac{-1}{x}\right) dx$

$$= -\frac{\ln(x)}{x} + \int \frac{1}{x^2}\,dx = \frac{-\ln(x)}{x} - \frac{1}{x} + C = \frac{-(\ln(x)+1)}{x} + C$$

$$\Rightarrow \int_1^e \frac{\ln(x)}{x^2}\,dx = -\frac{(\ln(x)+1)}{x}\Bigg|_1^e = \left(-\frac{(1+1)}{e}\right) - \left(-\frac{(0+1)}{1}\right)$$

$$= \frac{-2}{e} + 1$$

Alternate Strategy : Do everything in terms of definite integral.

Recall : If $F'(x) = f(x)$ and $u = g(x) \Rightarrow$

$$\frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)} \Rightarrow$$

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

$$\underset{F(g(x)) + C}{\shortparallel} \qquad\qquad \underset{F(u) + C}{\shortparallel}$$

$$\Rightarrow \int_a^b f(g(x))\,g'(x)\,dx = F(g(b)) - F(g(a)) = F(g(x))\Big|_a^b$$

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

Conclusion :

$$\boxed{\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$

Integration by Parts is easier :

$$\boxed{\int_a^b f(x)\,g(x)\,dx = f(x)\,G(x)\Big|_a^b - \int_a^b f'(x)\,G(x)\,dx}$$

The only reason to use these formulae is when we have to use geometric methods to determine a definite integral.

__Example__ $\displaystyle\int_0^{\sqrt{2}} 2x\sqrt{4-x^4}\,dx = ?$

__Method 1__ : Let $u = x^2 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow dx = \dfrac{du}{2x} \Rightarrow$

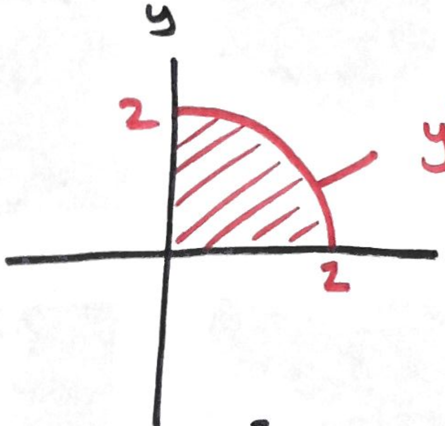
$$\int 2x\sqrt{4-x^4}\,dx = \int \sqrt{4-x^4}\,du = \int \sqrt{4-u^2}\,du$$

Problem : We cannot calculate $\int \sqrt{4-u^2}\, du$.

Method 2 : Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow$

$$\int_0^{\sqrt{2}} 2x \sqrt{4-x^4}\, dx = \int_0^{2} \sqrt{4-u^2}\, du$$

Remember to change limits of integration



$y^2+u^2 = 2^2 \Rightarrow y = \sqrt{4-u^2}$

$$\Rightarrow \int_0^{2} \sqrt{4-u^2}\, du = \text{Area}\left(\;\right) = \frac{1}{4} \cdot \pi \cdot 2^2 = \pi$$

$$\Rightarrow \int_0^{\sqrt{2}} 2x \sqrt{4-x^2}\, dx = \pi$$

Practical Example

A dam can hold water with depth at <u>most</u> 50 m. During a storm the depth of water increases at rate $500t^3e^{-t^2}$ m/hr where t = time in hours after 3 pm. Show that by 4pm the dam must be breached.

$h(t)$ = depth of water at time t (in hours after 3 pm) $\Rightarrow$

$h'(t) = 500t^3e^{-t^2}$ ($t = 0$ at 3 pm, $t = 1$ at 4 pm)

Depth increase between 4 pm and 3pm $= h(1) - h(0)$

$$= \int_0^1 500t^3e^{-t^2}\,dt.$$

Let $u = -t^2 \Rightarrow \frac{du}{dt} = -2t \Rightarrow dt = \frac{du}{-2t} \Rightarrow \int 500t^3e^{-t^2}\,dt$

$$= \int -250t^2e^{-t^2}\,du = 250\int ue^u\,du$$

$f(u) = u \quad g(u) = e^u$

$f'(u) = 1 \quad G(u) = e^u \Rightarrow \int ue^u\,du = ue^u - e^u + C$

$$\Rightarrow \int 500t^3e^{-t^2}\,dt = 250(-t^2-1)e^{-t^2} + C$$

$$\Rightarrow \int_0^1 500t^3e^{-t^2}\,dt = 250(-t^2-1)e^{-t^2}\Big|_0^1 = 250\left(1-\frac{2}{e}\right) \approx 66 > 50$$

$\Rightarrow$ The dam must be breached by 4pm.